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THE MAGNETIC INDUCTION OF THE SYSTEM CONSISTING OF A COIL AND A--ETC(U)
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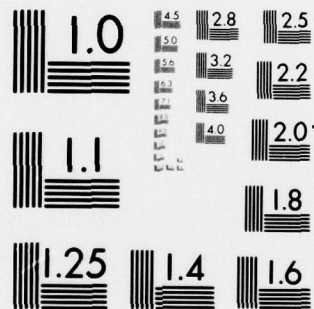
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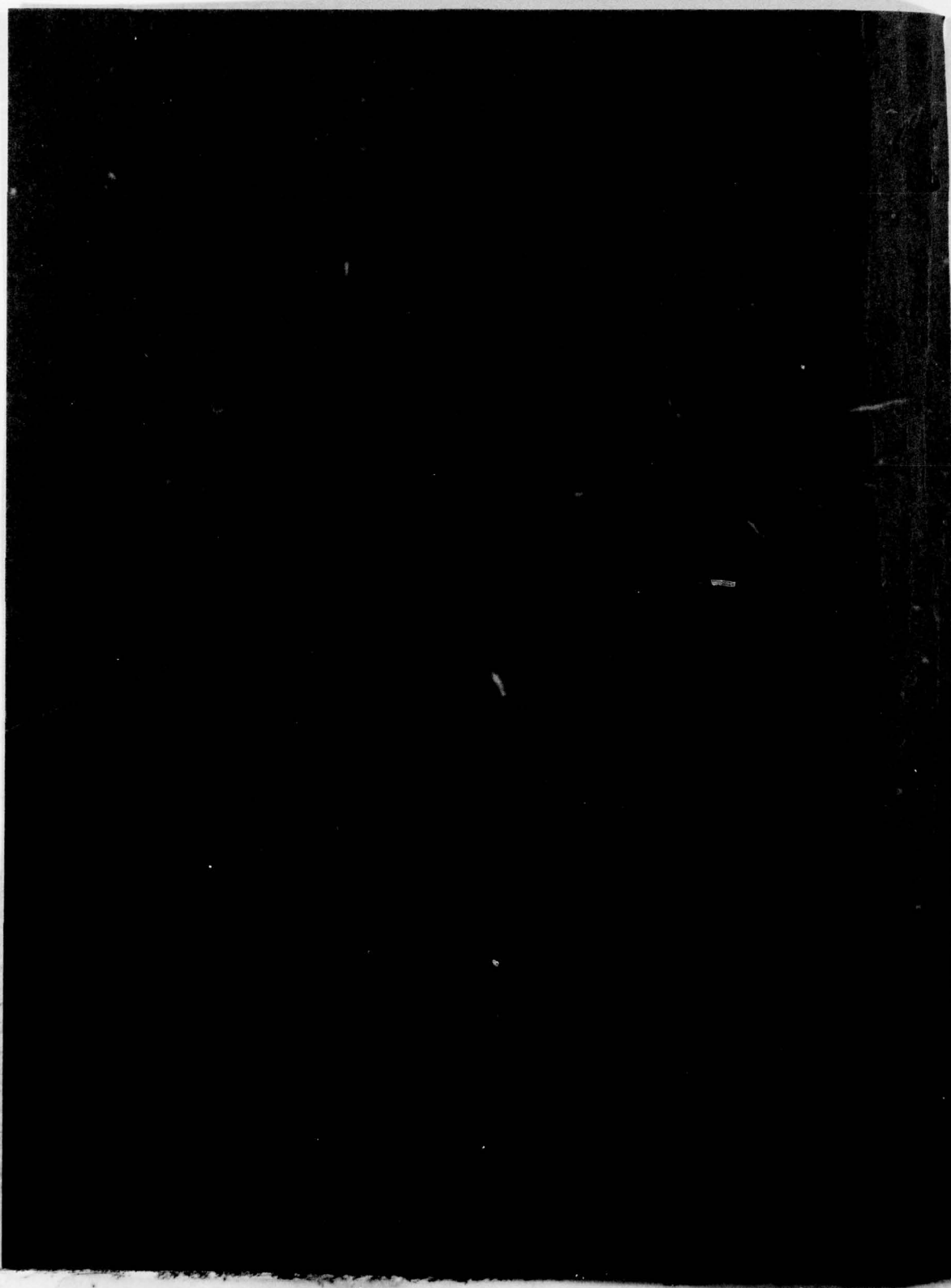
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LIST OF SYMBOLS

\bar{A}	Vector potential function
A_{ψ}	Psi (ψ) component of \bar{A}
$A_{\psi I}$	Psi component of \bar{A} in region I
$A_{\psi II}$	Psi component of \bar{A} in region II
$A_{\psi III}$	Psi component of \bar{A} in region III
$A_{\psi IV}$	Psi component of \bar{A} in region IV
\bar{B}	Magnetic flux density or magnetic induction
\bar{B}_1	Magnetic flux density in medium 1
\bar{B}_2	Magnetic flux density in medium 2
B_{η}	Eta component of the magnetic flux density
B_{θ}	Theta component of the magnetic flux density
$B_{\eta I}$	Eta component of \bar{B} in region I
$B_{\eta II}$	Eta component of \bar{B} in region II
$B_{\eta III}$	Eta component of \bar{B} in region III
$B_{\eta IV}$	Eta component of \bar{B} in region IV
B_{n1}	Normal component of \bar{B} in medium 1
B_{n2}	Normal component of \bar{B} in medium 2
\bar{H}	Magnetic field intensity
\bar{H}_1	Magnetic field intensity in medium 1
\bar{H}_2	Magnetic field intensity in medium 2
H_{t1}	Tangential component of \bar{H} in medium 1
H_{t2}	Tangential component of \bar{H} in medium 2

\bar{J}	Electric current density
J_η	Eta component of J
J_θ	Theta component of J
J_ψ	Psi component of J
\bar{J}_s	Surface current density
J	Magnitude of \bar{J}_s
χ_m	Magnetic susceptibility
μ	Magnetic permeability
μ_r	Relative magnetic permeability
μ_0	Permeability of free space
μ_1	Permeability
μ_2	Permeability
\bar{n}_{12}	Unit vector normal to interface; directed from medium 1 into medium 2
x, y, z	Rectangular coordinates
η, θ, ψ	Prolate spheroidal coordinates
\bar{e}_η	Unit normal vector in eta direction
\bar{e}_θ	Unit normal vector in theta direction
\bar{e}_ψ	Unit normal vector in azimuthal direction
$\bar{\nabla} \cdot$	Divergence operator
$\bar{\nabla} \times$	Curl operator
∇^2	Scalar Laplacian operation
∇^2	Vector Laplacian operator
p	Integer from one to infinity

$p_p^m(\cos\theta)$	Associated Legendre function of the first kind
$Q_p^m(\cos\theta)$	Associated Legendre function of the second kind
$\star \bar{A}_\psi$	Psi vector component of the vector Laplacian of \bar{A} in prolate spheroidal coordinates
$(\bar{\nabla} \times \bar{A})_\eta$	Eta component of the curl of A
$(\bar{\nabla} \times \bar{A})_\theta$	Theta component of the curl of A
θ_1, θ_2	Angles describing the limits of the current band
η_1, η_2, η_3	Constants
A_p, B_p, C_p	Constants in the general solution of $\star \bar{A}$ for current band outside the body where $p=1$ to ∞
D_p, E_p, F_p	
H_p, I_p, K_p	Constants in the general solution of $\star \bar{A}$ for the current band inside the body where $p=1$ to ∞
L_p, M_p, N_p	
ξ	Variable equal to $\cosh\eta$
ν	Variable equal to $\cos\theta$
e_1, e_2, e_3	Metric coefficients for a prolate spheroidal coordinate system
p_p^Δ	Variable use for simplification
Q_p^Δ	Variable use for simplification
C_1, C_2	Constants
K_1, K_2	
A', B'	
A, B	

G_p

Constants in current expansion where $p=1$ to ∞

$\star \bar{A}_\psi$

Psi vector component of the vector Laplacian of \bar{A} in prolate spheroidal coordinates

$$\left(= \frac{(\sinh^2 \eta + \sin^2 \theta)^{-1/2}}{a^2 (\sinh \eta \sin \theta)} \bar{e}_\psi \left(\frac{\partial}{\partial \eta} \frac{1}{\sinh \eta} \frac{\partial}{\partial \eta} (\sinh \eta A_\psi) \right. \right. \\ \left. \left. + \frac{\partial}{\partial \theta} \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_\psi) \right) \right)$$

$\star A_\psi$

Psi component of the vector Laplacian of \bar{A} in prolate spheroidal coordinates

$$\left(= \frac{(\sinh^2 \eta + \sin^2 \theta)^{-1/2}}{a^2 (\sinh \eta \sin \theta)} \left(\frac{\partial}{\partial \eta} \frac{1}{\sinh \eta} \frac{\partial}{\partial \eta} (\sinh \eta A_\psi) \right. \right. \\ \left. \left. + \frac{\partial}{\partial \theta} \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_\psi) \right) \right)$$

EXECUTIVE SUMMARY

THE MAGNETIC INDUCTION OF THE SYSTEM CONSISTING OF A COIL AND A FERROMAGNETIC PROLATE SPHEROIDAL SHELL

OBJECTIVE

The objective of this work was to derive solutions to static ferromagnetic problems that included both current-carrying coils and linear ferromagnetic bodies. The solutions are intended for comparison with solutions to ferromagnetic problems obtained by various numerical techniques such as the finite difference method, the finite element method, and the integral equation iterative solution method.

APPROACH

After deriving the governing differential equation from Maxwell's equations for classical magnetostatic field theory, the method of separation of variables was employed to obtain the problem solution.

RESULTS

The magnetic induction was calculated for two geometries (configurations) of a ferromagnetic prolate spheroidal shell and a current-carrying conductor. The first case was for an infinitesimally thin current band carrying a stationary current and surrounding the spheroidal shell. The second case was for a spheroidal shell surrounding an infinitesimally thin current band. The ferromagnetic bodies were assumed to be linear and homogeneous. The reduction of the solutions to that of a current band in free space is shown when the permeability of the ferromagnetic spheroidal shell is allowed to approach that of free space.

RECOMMENDATIONS

It is recommended that the derived solutions be programmed on a digital computer for direct comparison of these results to those obtained by various numerical methods. There are plans to implement these recommendations during the fiscal years 1979 and 1980.

ABSTRACT

The magnetic induction is calculated for two configurations consisting of: (1) an infinitesimally thin prolate spheroidal current band carrying a stationary current surrounding a ferromagnetic prolate spheroidal shell, and (2) an infinitesimally thin band internal to a ferromagnetic prolate spheroidal shell. The ferromagnetic body is assumed to be linear and homogeneous. The reduction of the solutions to that of a prolate spheroidal current band in free space is shown when the permeability of the ferromagnetic prolate spheroidal shell is allowed to approach that of free space.

ADMINISTRATIVE INFORMATION

This work was performed under Program Element 11221N, Project B0005, Task Area B0005-SL001, Work Unit 2704-110. The Project Director is Mr. W. J. Andahazy, David W. Taylor Naval Ship Research and Development Center.

INTRODUCTION

As noted in Brown and Baker's report,^{1*} exact analytical solutions of Maxwell's equations using classical formulations have been limited to body shapes and inhomogeneities that conform to a few separable coordinate systems. Modern digital computers with large computational and storage capabilities permit many electromagnetic field problems to be solved by using a numerical solution to the governing differential or integral equations under a suitable choice of boundary conditions. The numerical solutions of Maxwell's equations, when used with a complete description of the electric and magnetic sources and the constitutive laws of the media, can be used to describe completely the electric and magnetic fields produced by the source, including nonsymmetric geometries, nonsymmetric source distributions, and spatially varying media parameters.

*A list of references is given on page 71.

The motivation for this work arose out of the need for solutions to static ferromagnetic problems that could be used for comparison with numerical methods.

PROLATE SPHEROIDAL COORDINATE SYSTEM

The prolate spheroidal coordinate system can be formed by rotating the two-dimensional elliptic coordinate system, whose traces in a plane are confocal ellipses and hyperbolas, about the major axis of the ellipses.^{2,3}

Flammer³ notes that it is customary to make the z-axis the axis of revolution. Figure 1 depicts the three-dimensional prolate spheroidal coordinate system. In this case the coordinate surfaces are: prolate spheroids for $\eta = \text{constant}$; hyperboloids of two sheets for $\theta = \text{constant}$; meridian planes for $\psi = \text{constant}$. The prolate spheroidal coordinates shown on Figure 1 are related to rectangular coordinates by the following transformation equations:

$$x = a \sinh \eta \sin \theta \cos \psi \quad (1a)$$

$$y = a \sinh \eta \sin \theta \sin \psi \quad (1b)$$

$$z = a \cosh \eta \cos \theta \quad (1c)$$

where

$$0 \leq \eta < \infty$$

$$0 \leq \theta \leq \pi$$

$$0 \leq \psi < 2\pi$$

We have denoted the interfocal distance by $2a$ and the prolate spheroidal coordinates by (η, θ, ψ) .

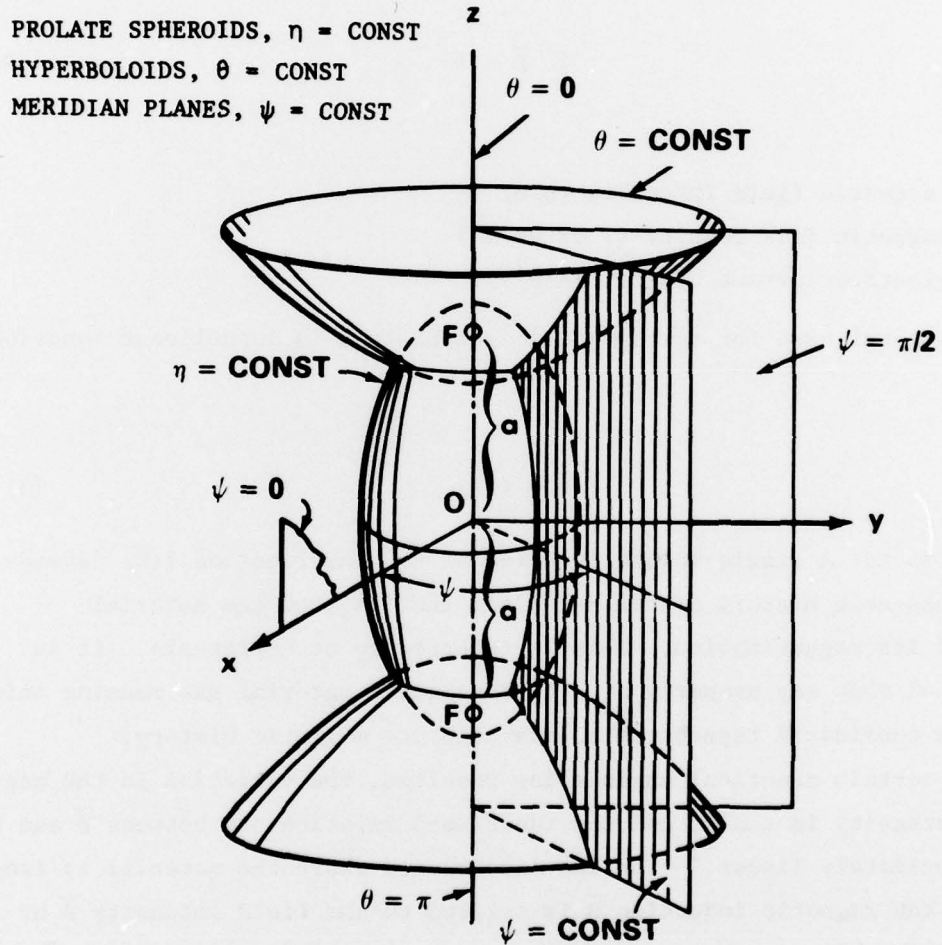


Figure 1 - Prolate Spheroidal Coordinate System

BASIC EQUATIONS

We can start with Maxwell's equations for classical magnetostatic field problems

$$\nabla \times \bar{H} = \bar{J} \quad (2a)^*$$

$$\nabla \cdot \bar{B} = 0 \quad (2b)$$

where

\bar{H} magnetic field intensity (A/m)

\bar{B} magnetic flux density (T or Wb/m²)

\bar{J} electric current density (A/m²) .

In the general case for ferromagnetic materials \bar{B} is a nonlinear function of \bar{H}

$$\bar{B} = f(\bar{H}) \quad (3)$$

where \bar{B} is not a single valued function of \bar{H} . The function $f(\bar{H})$ depends on the magnetic history of the material, that is, how the material attained its magnetization. This is referred to as hysteresis. It is also noted that any property of a ferromagnetic material has meaning only if it is considered together with its complete magnetic history.

In certain practical engineering problems, the variation in the magnetic intensity is small, and the functional relationship between \bar{B} and \bar{H} is approximately linear.¹ For the linear case where the material is isotropic, the magnetic induction \bar{B} is related to the field intensity \bar{H} by the relationship

*The del operator ∇ is defined with respect to the rectangular coordinate system and is strictly valid in a rectangular coordinate system only. Very often $\nabla \times$ and $\nabla \cdot$ are used as equivalent symbols for curl and divergence generally. This use is followed in this report.

$$\vec{B} = \mu_0 (\chi_m + 1) \vec{H} = \mu_0 \mu_r \vec{H} = \mu \vec{H} \quad (4)$$

where

χ_m magnetic susceptibility (dimensionless)

μ magnetic permeability (henry/meter)

$(\chi_m + 1) = \mu_r$ relative permeability (dimensionless)

μ_0 free space permeability ($4\pi \times 10^{-7}$ henry/meter).

This report assumes that the ferromagnetic body has isotropic and linear material properties. The divergenceless nature of the magnetic flux density in conjunction with the fact that the divergence of the curl of any vector function is zero allows the introduction of the magnetic vector potential field (\vec{A})

$$\vec{B} = \nabla \times \vec{A} \quad (5)$$

where \vec{A} is the magnetostatic vector potential function in weber/meter. The substitution of Equation (5) into Equation (2a) gives the fundamental equation of the vector potential of the magnetostatic field.

$$\frac{1}{\mu} \nabla \times (\nabla \times \vec{A}) - (\nabla \times \vec{A}) \times \nabla \frac{1}{\mu} = \vec{J} \quad (6)$$

For homogeneous materials, as assumed in this report, the magnetic permeability is spatially invariant. Hence,

$$\nabla \frac{1}{\mu} = 0 \quad (7)$$

and Equation (6) reduces to

$$\bar{\nabla} \times (\bar{\nabla} \times \bar{A}) = \mu \bar{J} \quad (8)$$

Using the vector identity

$$\bar{\nabla} \times \bar{\nabla} \times \bar{A} = \bar{\nabla}(\bar{\nabla} \cdot \bar{A}) - \nabla^2 \bar{A} \quad (9)^*$$

Equation 8 becomes

$$\bar{\nabla}(\bar{\nabla} \cdot \bar{A}) - \nabla^2 \bar{A} = \mu \bar{J} \quad (10)$$

The magnetostatic vector potential is characterized by the important property that its divergence can be conveniently chosen to be zero.

$$\bar{\nabla} \cdot \bar{A} = 0 \quad (11)$$

Equation (10) reduces to the vector Poisson's differential equation.

$$\nabla^2 \bar{A} = -\mu \bar{J} \quad (12)$$

This is the governing equation for our calculations.

The general boundary conditions to be satisfied at the interfaces of stationary dissimilar media may be derived from the limiting integral forms of Maxwell's equations and are given by

*The vector Laplacian operator is designated by ∇^2 .

$$\bar{n}_{12} \cdot (\bar{B}_2 - \bar{B}_1) = 0 \text{ or } B_{n1} = B_{n2} \quad (13a)$$

$$\bar{n}_{12} \times (\bar{H}_2 - \bar{H}_1) = \bar{J}_s \text{ or } H_{t2} - H_{t1} = J_s \quad (13b)$$

where the subscripts 1 and 2 indicate the media under consideration, and \bar{n}_{12} denotes the unit normal vector to the interface and is directed from medium 1 into medium 2. In the case where the materials are linear and isotropic equations (13a) and (13b) become

$$\bar{n}_{12} \cdot (\mu_2 \bar{H}_2 - \mu_1 \bar{H}_1) = 0 \quad (13c)$$

$$\bar{n}_{12} \times \left(\frac{\bar{B}_2}{\mu_2} - \frac{\bar{B}_1}{\mu_1} \right) = \bar{J}_s \quad (13d)$$

\bar{J}_s is a true surface current density that may exist at the interface. At an interface where \bar{J}_s is 0, Equations (13b) and (13d) need to be modified accordingly.

THIN COIL SURROUNDING A FERROMAGNETIC PROLATE SPHEROIDAL SHELL

GENERAL SOLUTION

We now proceed to solve the boundary value problem of a ferromagnetic prolate spheroidal shell of homogeneous permeability μ_2 surrounded by an infinitesimally thin prolate spheroidal current band of constant current density \bar{J} . The geometry of the problem suggests that a prolate spheroidal coordinate system as shown in Figure 1 be used in the solution.

Figure 2, a cross section of the problem geometry, identifies the four regions of interest. The boundaries of the prolate spheroidal shell are

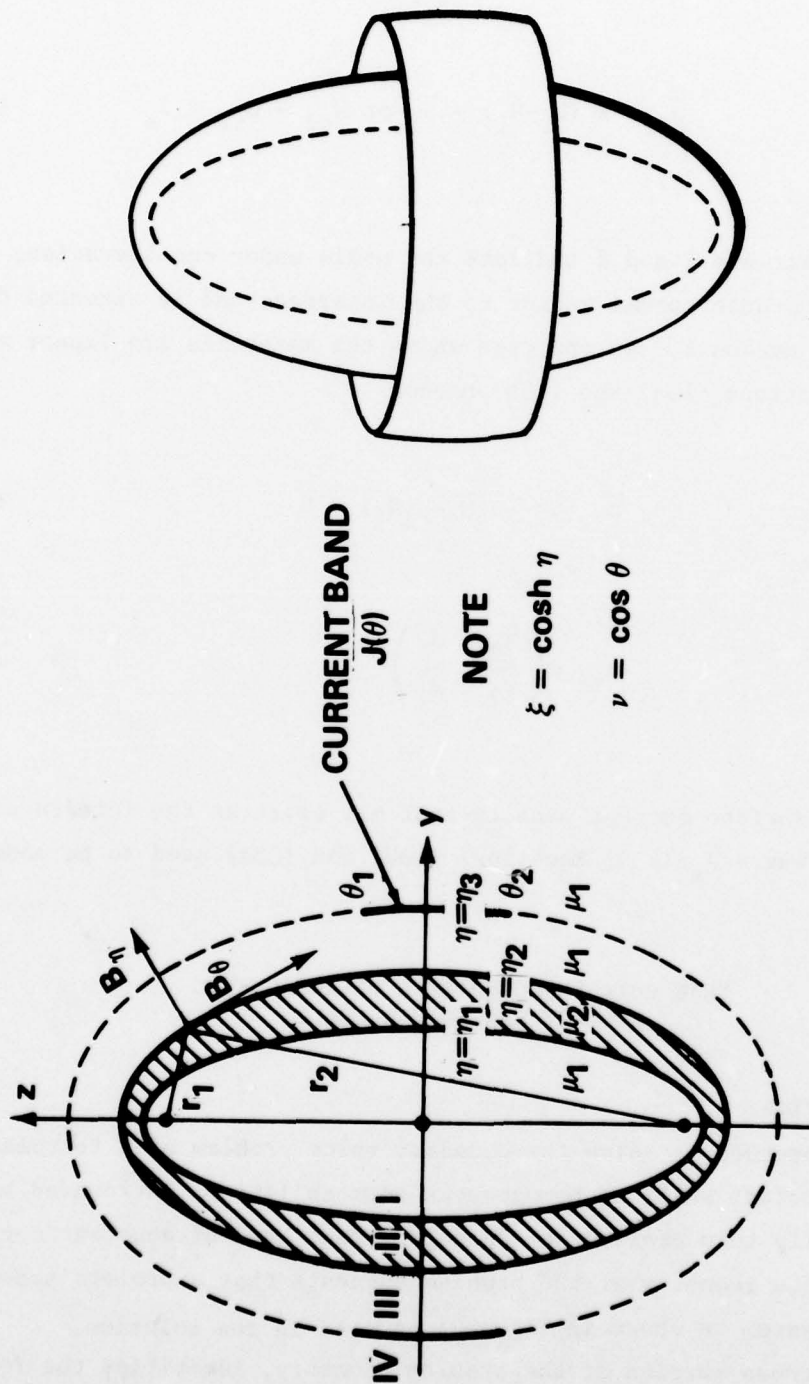


Figure 2 - Cross Section of Ferromagnetic Spheroidal Shell Surrounded by Infinitesimally Thin Current Band

determined by $\eta = \eta_1 = \text{constant}$ and $\eta = \eta_2 = \text{constant}$. The dc current lies in the boundary $\eta = \eta_3$ constant. Regions I, III, and IV have a permeability equal to that of free space μ_0 , which for convenience will be labelled μ_1 . Ampere's law states

$$\nabla \times \bar{H} = \bar{J} \quad (14)$$

and since $\nabla \cdot \bar{B} = 0$, the induction \bar{B} must be the curl of some vector field \bar{A} . The governing differential equation for \bar{A} when homogeneous and linear materials are considered is from Equation (12)

$$\nabla^2 \bar{A} = -\mu \bar{J} \quad (15)$$

The general expression in prolate spheroidal coordinates for a current density is

$$\bar{J} = J_\eta \bar{e}_\eta + J_\theta \bar{e}_\theta + J_\psi \bar{e}_\psi \quad (16)$$

In the problem presented herein, the current density has only a psi (ψ) component $[J_\psi(\theta) \bar{e}_\psi]$ which means that the vector potential has only a psi component $A_\psi \bar{e}_\psi$. The vector potential $\bar{A} = A_\psi \bar{e}_\psi$ is a function of the prolate spheroidal coordinates η, θ [i.e. $A_\psi = A_\psi(\eta, \theta)$]. The constant current density, which lies on the boundary between regions III and IV, can be expressed by the function

$$\bar{J} = \begin{cases} 0 & , \text{ if } \theta < \theta_1 \text{ or } \theta > \theta_2 \\ J_\psi(\theta) \bar{e}_\psi & , \text{ if } \theta_1 \leq \theta \leq \theta_2 \end{cases} \quad (17)$$

where $J_\psi(\theta)$ is equal to a constant J along $\eta = \eta_3$ for $\theta_1 \leq \theta \leq \theta_2$.

Therefore, Equation (15) has only an azimuthal component and can be expressed as

$$\nabla^2 \bar{A}_\psi = \nabla^2 A_\psi(\eta, \theta) = 0 \quad \left(\begin{array}{l} \text{in regions I} \\ \text{through IV} \end{array} \right) \quad (18)$$

When the psi component of vector Laplacian $\nabla^2 A_\psi$ is expanded in prolate spheroidal coordinates, Equation (18) can be expressed as (see Appendix A)

$$\frac{\partial}{\partial \eta} \left[\frac{1}{\sinh \eta} \frac{\partial(\sinh \eta A_\psi)}{\partial \eta} \right] + \frac{\partial}{\partial \theta} \left[\frac{1}{\sin \theta} \frac{\partial(\sin \theta A_\psi)}{\partial \theta} \right] = 0 \quad (19)$$

(in regions I-IV)

Applying the method of separation of variables, let us assume that A_ψ can be expressed as the product of two functions

$$A_\psi = H(\cosh \eta) G(\cos \theta) \quad (20)$$

where $H(\cosh \eta)$ is a function of $\cosh \eta$ only and $G(\cos \theta)$ is a function of $\cos \theta$ only. Substituting this form of the component of the vector potential \bar{A} into Equation (19), we have after separation of variables

$$\frac{d^2 H}{d\eta^2} + \coth \eta \frac{dH}{d\eta} - \left(p(p+1) + \frac{1}{\sinh^2 \eta} \right) H = 0 \quad (21a)$$

$$\frac{d^2 G}{d\theta^2} + \cot \theta \frac{dG}{d\theta} + \left(p(p+1) - \frac{1}{\sin^2 \theta} \right) G = 0 \quad (21b)$$

where the separation constant is $p(p+1)$ and p is an integer from one to infinity. It is well known that differential equations of the form

$$\frac{d^2 H'}{d\eta^2} + \coth\eta \frac{dH'}{d\eta} - \left(p(p+1) + \frac{m^2}{\sinh^2 \eta} \right) H' = 0 \quad (22a)$$

have the general solution of the form

$$H' = C_1 P_p^m (\cosh\eta) + C_2 Q_p^m (\cosh\eta) \quad (22b)$$

where C_1 and C_2 are constants, and it is known that a differential equation of the form

$$\frac{d^2 G'}{d\theta^2} + \cot\theta \frac{dG'}{d\theta} + \left[p(p+1) - \frac{m^2}{\sin^2 \theta} \right] G' = 0 \quad (23a)$$

has the general solution of the type

$$G' = C_3 P_p^m (\cos\theta) + C_4 Q_p^m (\cos\theta) \quad (23b)$$

where C_3 and C_4 are constants. P_p^m and Q_p^m are the associated Legendre functions of the first and second kind, respectively. Comparison of Equations (21), (22), and (23) shows that in Equations (22) and (23), m^2 is equal to 1. This requires that m always equal unity. The solutions of Equations (21a) and (21b) are expressed as

$$H(\cosh\eta) = A P_p^1 (\cosh\eta) + B Q_p^1 (\cosh\eta) \quad (24a)$$

$$G(\cos\theta) = A' P_p^1(\cos\theta) + B' Q_p^1(\cos\theta) \quad (24b)$$

The general solution of Equation (19) may be formed from the product of solutions in Equations (24a) and (24b) which yield

$$A_\psi = H(\cosh\eta) G(\cos\theta) = \sum_{p=1}^{\infty} H_p(\cosh\eta) G_p(\cos\theta) \quad (25)$$

$$A_\psi = \sum_{p=1}^{\infty} \left[A P_p^1(\cosh\eta) + B Q_p^1(\cosh\eta) \right] \times \left[A' P_p^1(\cos\theta) + B' Q_p^1(\cos\theta) \right] \quad (26)$$

For the prolate spheroidal system, the associated Legendre functions of the second kind are infinite at $\cos\theta = \pm 1$ and as such cannot be included in a general solution for a given region which includes $\theta = 0$ or $\theta = \pi$. Therefore, in our case the constant B' is set equal to zero. Equation (26) reduces to

$$A_\psi = \sum_{p=1}^{\infty} \left[K_1 P_p^1(\cosh\eta) + K_2 Q_p^1(\cosh\eta) \right] P_p^1(\cos\theta) \quad (27)$$

where K_1 and K_2 are constants ($K_1 = AA'$, $K_2 = AB'$). When the substitutions $\xi = \cosh\eta$ and $v = \cos\theta$ are made in Equation (27), A_ψ can be expressed as

$$A_\psi = \sum_{p=1}^{\infty} \left[K_1 P_p^1(\xi) + K_2 Q_p^1(\xi) \right] P_p^1(v) \quad (28)$$

This is the general form of the psi component of the vector potential that will be used to determine the potentials A_ψ in each region.

BOUNDARY CONDITIONS

The form of the component of the vector potential A_ψ in regions (I-IV) is determined from Equation (28). These magnetostatic vector potentials in regions I-IV are:

$$\begin{aligned} A_{\psi I} &= \sum_{p=1}^{\infty} \left[A_p P_p^1(\xi) \right] P_p^1(\nu) \\ A_{\psi II} &= \sum_{p=1}^{\infty} \left[B_p P_p^1(\xi) + C_p Q_p^1(\xi) \right] P_p^1(\nu) \\ A_{\psi III} &= \sum_{p=1}^{\infty} \left[D_p P_p^1(\xi) + E_p Q_p^1(\xi) \right] P_p^1(\nu) \\ A_{\psi IV} &= \sum_{p=1}^{\infty} \left[F_p Q_p^1(\xi) \right] P_p^1(\nu) \end{aligned} \tag{29}$$

Because the potential must be finite in each of the regions I, II, and III and approach zero as $\xi \rightarrow \infty$ in region IV, the following constants were set equal to zero.

a. For $A_{\psi I}$ the constant associated with $Q_p^1(\xi) P_p^1(\nu)$ was set equal to zero because

$$Q_p^1(\xi) \rightarrow \infty \text{ at } \xi = +1$$

b. For $A_{\psi IV}$ the constant associated with $P_p^1(\xi) P_p^1(\nu)$ was set equal to zero because $P_p^1(\xi) \rightarrow \infty$ as $\xi \rightarrow \infty$.

(We note $Q_p^1(\xi) \rightarrow 0$ as $\xi \rightarrow \infty$)

The constants $A_p, B_p, C_p, D_p, E_p,$ and F_p are to be determined from the boundary conditions. At each interface, the basic laws of magnetostatics (Equations (2a) and (2b)) reduce to boundary conditions on \bar{B} and \bar{H} that can be used to evaluate these six constants. The normal component of \bar{B} across each boundary must be continuous, i.e., $(\bar{B}_2 - \bar{B}_1) \cdot \bar{n}_{12} = 0$ where the quantity \bar{n}_{12} is the unit outward normal to the surface. This provides the following boundary conditions which must be satisfied by the solutions given in Equation (29) for each region.

$$B_{\eta I} = B_{\eta II} \text{ at } \eta = \eta_1 \quad (30a)$$

$$B_{\eta II} = B_{\eta III} \text{ at } \eta = \eta_3 \quad (30b)$$

$$B_{\eta III} = B_{\eta IV} \text{ at } \eta = \eta_3 \quad (30c)$$

The eta or normal component of the magnetic field (B_η) is expressed in terms of the vector potential as

$$\begin{aligned} B_\eta &= (\bar{\nabla} \times \bar{A}_\psi)_\eta = \frac{1}{e_2 e_3} \frac{\partial(e_3 A_\psi)}{\partial \theta} \\ &= - \frac{1}{a(\xi^2 - v^2)^{1/2}} \frac{\partial}{\partial v} \left[(1 - v^2)^{1/2} A_\psi \right] \end{aligned} \quad (31)$$

where

$$\bar{B} = \bar{\nabla} \times \bar{A}$$

$$= \frac{1}{a(\sinh^2 \eta + \sin^2 \theta) (\sinh \eta \sin \theta)} \times$$

$$\begin{vmatrix} \bar{e}_\eta (\sinh^2 \eta + \sin^2 \theta)^{1/2} & \bar{e}_\theta (\sinh^2 \eta + \sin^2 \theta)^{1/2} & \bar{e}_\psi \sinh \eta \sin \theta \\ \frac{\partial}{\partial \eta} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \psi} \\ 0 & 0 & A_\psi \sinh \eta \sin \theta \end{vmatrix}$$

and

$$\xi = \cosh \eta, e_1 = e_2 = a(\sinh^2 \eta + \sin^2 \theta)^{1/2} = a(\xi^2 - v^2)^{1/2}$$

$$v = \cos \theta, e_3 = a \sinh \eta \sin \theta$$

However, since the vector potentials in each region are functions of $P_p^1(v)$, we can simplify Equation (30) to constraints on A_ψ at the interfaces:

$$A_{\psi I} = A_{\psi II} \text{ at } \eta = \eta_1 \quad (32a)$$

$$A_{\psi II} = A_{\psi III} \text{ at } \eta = \eta_2 \quad (32b)$$

$$A_{\psi III} = A_{\psi IV} \text{ at } \eta = \eta_3 \quad (32c)$$

The second set of boundary conditions states that the theta or tangential component of \bar{H} across each boundary must satisfy the relationship

$$\bar{n}_{12} \times (\bar{H}_2 - \bar{H}_1) = \bar{J}_s \quad (33)$$

where J_s (which equals $J_\psi(\theta)\bar{e}_\psi$) is the real surface current density in the limit of vanishing width between the two regions. Using the relationship $\bar{B} = \mu\bar{H}$, Equation (33) can be expressed as

$$\frac{B_{\theta 2}}{\mu_2} - \frac{B_{\theta 1}}{\mu_1} = J_\psi(\theta) \quad (34)$$

Referring to the curl in Equation (31), we can write B_θ in the form

$$B_\theta = (\bar{\nabla} \times \bar{A}_\psi)_\theta \quad (35)$$

$$= \frac{-1}{e_1 e_3} \frac{\partial(e_3 A_\psi)}{\partial \eta} = - \frac{1}{a(\xi_1^2 - v^2)^{1/2}} \frac{\partial}{\partial \xi} \left[(\xi^2 - 1)^{1/2} A_\psi \right]$$

From Equations (34) and (35) the tangential components of \bar{B} in regions I-IV must satisfy the relationships:

$$\left(\frac{1}{\mu_2} \right) \frac{1}{a(\xi_1^2 - v^2)^{1/2}} \frac{\partial}{\partial \xi} \left[(\xi^2 - 1)^{1/2} A_{\psi II} \right] \bigg|_{\xi=\xi_1} = \quad (36a)$$

$$\left(\frac{1}{\mu_1} \right) \frac{1}{a(\xi_1^2 - v^2)^{1/2}} \frac{\partial}{\partial \xi} \left[(\xi^2 - 1)^{1/2} A_{\psi I} \right] \bigg|_{\xi=\xi_1}$$

$$\left(\frac{1}{\mu_1} \right) \frac{1}{a(\xi_2^2 - \nu^2)^{1/2}} \frac{\partial}{\partial \xi} \left[(\xi^2 - 1)^{1/2} A_{\psi \text{III}} \right] \Big|_{\xi=\xi_2} = \quad (36b)$$

$$\begin{aligned} & \left(\frac{1}{\mu_2} \right) \frac{1}{a(\xi_2^2 - \nu^2)^{1/2}} \frac{\partial}{\partial \xi} \left[(\xi^2 - 1)^{1/2} A_{\psi \text{II}} \right] \Big|_{\xi=\xi_2} \\ & - \left(\frac{1}{\mu_1} \right) \frac{1}{a(\xi_3^2 - \nu^2)^{1/2}} \frac{\partial}{\partial \xi} \left[(\xi^2 - 1)^{1/2} A_{\psi \text{IV}} \right] \Big|_{\xi=\xi_3} \\ & + \left(\frac{1}{\mu_1} \right) \frac{1}{a(\xi_3^2 - \nu^2)^{1/2}} \frac{\partial}{\partial \xi} \left[(\xi^2 - 1)^{1/2} A_{\psi \text{III}} \right] \Big|_{\xi=\xi_3} = \quad (36c) \end{aligned}$$

$$\sum_p J_p(\theta) = \sum_p \frac{K G P_p^1(\nu)}{a(\xi_3^2 - \nu^2)^{1/2}}$$

It is understood that the summation runs from $p = 1$ to $p = \infty$.

The general expressions for the potentials in each region (Equation (29)) are then substituted into the boundary conditions (Equations (32) and (36)) are solved for the six constants (A_p , B_p , C_p , D_p , E_p , and F_p). Since there are six equations with six unknowns, the potential in each region can be specified. The six boundary value equations are presented below. The index p in the summation sign has both even and odd values and takes on

values from 1 to ∞ . It is noted at this point that the current density $J_\psi(\theta)$ must be expanded into a set of associated Legendre functions in order to evaluate the constants in the vector potential (Equation (29)). The detailed expansion is presented in Appendix B. The six expressions for the boundary conditions are:

$$A_P P_P^1(\xi_1) P_P^1(\nu) = \left[B_P P_P^1(\xi_1) + C_P Q_P^1(\xi_1) \right] P_P^1(\nu) \quad (37a)$$

$$\left(\frac{1}{\mu_2} \right) \frac{1}{a(\xi_1^2 - \nu^2)^{1/2}} \frac{\partial}{\partial \xi} \left[(\xi^2 - 1)^{1/2} (B_P P_P^1(\xi) + C_P Q_P^1(\xi)) P_P^1(\nu) \right] \Big|_{\xi=\xi_1} = \quad (37b)$$

$$\left(\frac{1}{\mu_1} \right) \frac{1}{a(\xi_1^2 - \nu^2)^{1/2}} \frac{\partial}{\partial \xi} \left[(\xi^2 - 1)^{1/2} (A_P P_P^1(\xi)) P_P^1(\nu) \right] \Big|_{\xi=\xi_1}$$

$$\left[B_P P_P^1(\xi_2) + C_P Q_P^1(\xi_2) \right] P_P^1(\nu) = \left[D_P P_P^1(\xi_2) + E_P Q_P^1(\xi_2) \right] P_P^1(\nu) \quad (37c)$$

$$\left(\frac{1}{\mu_1} \right) \frac{1}{a(\xi_2^2 - \nu^2)^{1/2}} \frac{\partial}{\partial \xi} \left[(\xi^2 - 1)^{1/2} (D_P P_P^1(\xi) + E_P Q_P^1(\xi)) P_P^1(\nu) \right] \Big|_{\xi=\xi_2} = \quad (37d)$$

$$\left(\frac{1}{\mu_2} \right) \frac{1}{a(\xi_2^2 - \nu^2)^{1/2}} \frac{\partial}{\partial \xi} \left[(\xi^2 - 1)^{1/2} (B_P P_P^1(\xi) + C_P Q_P^1(\xi)) P_P^1(\nu) \right] \Big|_{\xi=\xi_2}$$

$$\left[D_p P_p^1(\xi_3) + E_p Q_p^1(\xi_3) \right] P_p^1(v) = F_p Q_p^1(\xi_3) P_p^1(v) \quad (37e)$$

$$- \left(\frac{1}{\mu_1} \right) \frac{1}{a(\xi_3^2 - v^2)^{1/2}} \frac{\partial}{\partial \xi} \left[(\xi^2 - 1)^{1/2} \left(F_p Q_p^1(\xi) \right) P_p^1(v) \right] \Bigg|_{\xi=\xi_3} +$$

$$\left(\frac{1}{\mu_1} \right) \frac{1}{a(\xi_3^2 - v^2)^{1/2}} \frac{\partial}{\partial \xi} \left[(\xi^2 - 1)^{1/2} \left(D_p P_p^1(\xi) + E_p Q_p^1(\xi) \right) P_p^1(v) \right] \Bigg|_{\xi=\xi_3} = \quad (37f)$$

$$J_p(\theta) = \frac{KG P_p^1(v)}{a(\xi_3^2 - v^2)^{1/2}}$$

If we make the following substitution

$$P_p^\Delta(\xi) = \frac{d}{d\xi} \left[(\xi^2 - 1)^{1/2} P_p^1(\xi) \right] \quad (38)$$

$$Q_p^\Delta(\xi) = \frac{d}{d\xi} \left[(\xi^2 - 1)^{1/2} Q_p^1(\xi) \right] \quad (39)$$

and perform simple algebraic manipulations, the six boundary conditions can be simplified to:

$$A_p P_p^1(\xi_1) = B_p P_p^1(\xi_1) + C_p Q_p^1(\xi_1) \quad (40a)$$

$$\left(\frac{1}{\mu_2}\right) \left[B_p P_p^\Delta(\xi_1) + C_p Q_p^\Delta(\xi_1) \right] = \left(\frac{1}{\mu_1}\right) A_p P_p^\Delta(\xi_1) \quad (40b)$$

$$B_p P_p^1(\xi_2) + C_p Q_p^1(\xi_2) = D_p P_p^1(\xi_2) + E_p Q_p^1(\xi_2) \quad (40c)$$

$$\left(\frac{1}{\mu_1}\right) \left[D_p P_p^\Delta(\xi_2) + E_p Q_p^\Delta(\xi_2) \right] = \left(\frac{1}{\mu_2}\right) \left[B_p P_p^\Delta(\xi_2) + C_p Q_p^\Delta(\xi_2) \right] \quad (40d)$$

$$D_p P_p^1(\xi_3) + E_p Q_p^1(\xi_3) = F_p Q_p^1(\xi_3) \quad (40e)$$

$$-\left(\frac{1}{\mu_1}\right) F_p Q_p^\Delta(\xi_3) + \left(\frac{1}{\mu_1}\right) D_p P_p^\Delta(\xi_3) + \left(\frac{1}{\mu_1}\right) E_p Q_p^\Delta(\xi_3) = \frac{J_p(\theta) a(\xi_3^2 - v^2)^{1/2}}{P_p^1(v)} \quad (40f)$$

where

$$J_p(\theta) = \frac{K G P_p^1(v)}{a(\xi_3^2 - v^2)^{1/2}}$$

The solution of these six simultaneous equations to obtain E_p in terms of known quantities gives:

$$E_p = \frac{-\frac{1}{\mu_2} \left([x] J_p^{II} P_p^\Delta(\xi_2) \right) - \frac{1}{\mu_2} \left(J_p^{II} \right) Q_p^\Delta(\xi_2) + \frac{1}{\mu_1} \left(J_p^I \right) P_p^\Delta(\xi_2)}{\frac{1}{\mu_2} \left([x] [z] P_p^\Delta(\xi_2) \right) + \frac{1}{\mu_2} \left([z] Q_p^\Delta(\xi_2) \right) - \left(\frac{1}{\mu_1} \right) Q_p^\Delta(\xi_2)} \quad (41a)$$

where

$$[x] = \frac{\left(\left(\frac{\mu_1}{\mu_2} \right) \frac{Q_p^\Delta(\xi_1)}{P_p^\Delta(\xi_1)} - \frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} \right)}{\left(1 - \frac{\mu_1}{\mu_2} \right)} \quad (41b)$$

$$[z] = \frac{Q_p^1(\xi_2)}{[x]P_p^1(\xi_2) + Q_p^1(\xi_2)} \quad (41c)$$

$$J_p(\theta) = \frac{K G P_p^1(v)}{a(\xi_3^2 - v^2)^{1/2}}, \quad (v = \cos\theta) \quad (41d)$$

$$J_p^I = \left\{ \frac{-\mu_1 J_p(\theta) a(\xi_3^2 - v^2)^{1/2}}{P_p^1(v) Q_p^\Delta(\xi_3)} \right\} / \left(\frac{P_p^1(\xi_3)}{Q_p^1(\xi_3)} - \frac{P_p^\Delta(\xi_3)}{Q_p^\Delta(\xi_3)} \right) \quad (41e)$$

$$J_p^{II} = \frac{J_p^I P_p^1(\xi_2)}{[x]P_p^1(\xi_2) + Q_p^1(\xi_2)} \quad (41f)$$

The numerical values for the other five coefficients can be obtained from the following equations:

$$C_p = J_p^{II} + E_p[z] \quad (42a)$$

$$B_p = [x]C_p \quad (42b)$$

$$D_p = J_p^I \quad (42c)$$

$$A_p = B_p + C_p \frac{C_p^1(\xi_1)}{P_p^1(\xi_1)} \quad (42d)$$

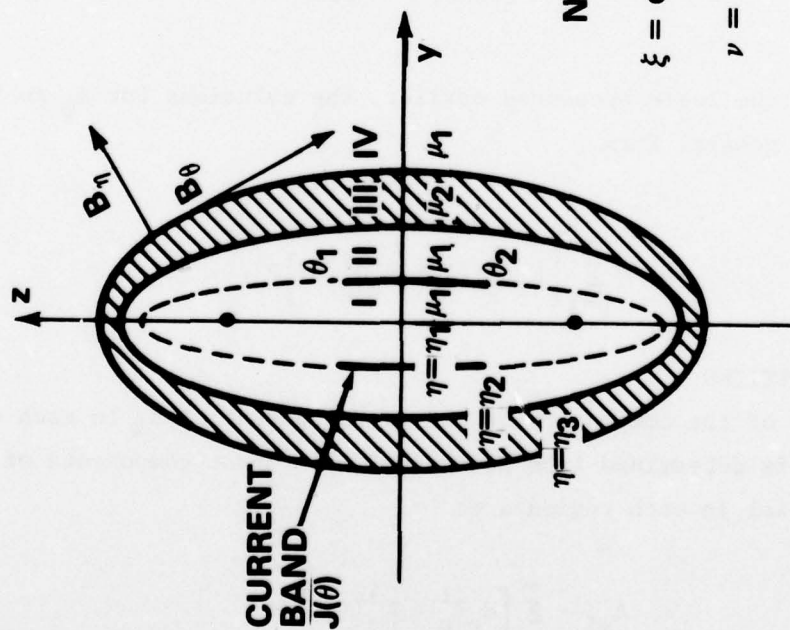
$$F_p = \frac{D_p P_p^1(\xi_3)}{Q_p^1(\xi_3)} + E_p \quad (42e)$$

Since the six coefficients can be determined for a specified problem from Equations (41) and (42), the potentials $A_{\psi I}$, $A_{\psi II}$, $A_{\psi III}$, and $A_{\psi IV}$, in regions I through IV can be completely determined. The normal (B_η) and tangential (B_θ) to the surface $\eta = \text{constant}$ (or $\xi = \text{constant}$) components of the magnetic induction in each region I through IV can be determined by using Equations (31) and (35).

THIN COIL INTERNAL TO A FERROMAGNETIC PROLATE SPHEROIDAL SHELL

GENERAL SOLUTION

We now proceed to solve the boundary value problem of a ferromagnetic spheroidal shell of homogeneous permeability μ_2 , surrounding an infinitesimally thin prolate spheroidal current band having a constant current density \bar{J} . Figure 3 shows the cross section of the problem geometry. The coordinate system shown previously in Figure 1 will be used in the solution. The boundaries of the prolate spheroidal shell are determined by $\eta = \eta_3$ and $\eta = \eta_2$. The steady state current lies in the boundary $\eta = \eta_1$ and between $\theta_1 \leq \theta \leq \theta_2$. As in the previous problem, the dc current density has only a psi component $J_\psi(\theta)\bar{e}_\psi$, and thus the vector potential has only a psi component $A_\psi\bar{e}_\psi$. The vector potential is a function of the prolate spheroidal coordinates η and θ . The constant current density is expressed by Equation (17) when the boundary η is changed to $\eta = \eta_1$.



Note

$$\xi = \cosh \eta$$

$$\nu = \cos \theta$$

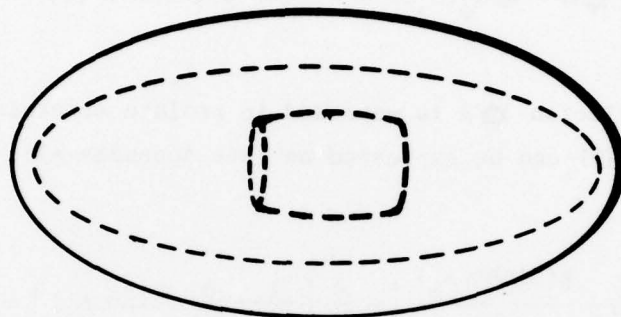


Figure 3 - Cross Section of Infinitesimally Thin Current Band Surrounded by Ferromagnetic Spheroidal Shell

The governing partial differential equation has only a psi component and is given by

$$\nabla^2 \bar{A} = \nabla^2 \bar{A}_\psi(\eta, \theta) = 0 \quad (\text{in regions I-IV}) \quad (43)$$

When the vector Laplacian $\nabla^2 A$ is expanded in prolate spheroidal coordinates, Equation (43) can be expressed as (see Appendix A):

$$\frac{\partial}{\partial \eta} \left[\frac{1}{\sinh \eta} \frac{\partial (\sinh \eta A_\psi)}{\partial \eta} \right] + \frac{\partial}{\partial \theta} \left[\frac{1}{\sin \theta} \frac{\partial (\sin \theta A_\psi)}{\partial \theta} \right] = 0$$

Adopting the following notation

$$\xi = \cosh \eta, \quad v = \cos \theta \quad (45)$$

and following the logic presented earlier, the solutions for A_ψ in regions I-IV have the general form

$$A = \sum_{p=1}^{\infty} \left[K_1 P_p^1(\xi) + K_2 Q_p^1(\xi) \right] P_p^1(v) \quad (46)$$

BOUNDARY CONDITIONS

The form of the components of the vector potential A_ψ in each of the regions I-IV is determined from Equation (46). These components of the vector potential in each region are:

$$A_{\psi I} = \sum_{p=1}^{\infty} \left[H_p P_p^1(\xi) \right] P_p^1(v) \quad (47a)$$

$$A_{\psi II} = \sum_{p=1}^{\infty} \left[I_p P_p^1(\xi) + K_p Q_p^1(\xi) \right] P_p^1(v) \quad (47b)$$

$$A_{\psi III} = \sum_{p=1}^{\infty} \left[L_p P_p^1(\xi) + M_p Q_p^1(\xi) \right] P_p^1(\nu) \quad (47c)$$

$$A_{\psi IV} = \sum_{p=1}^{\infty} \left[N_p Q_p^1(\xi) \right] P_p^1(\nu) \quad (47d)$$

The P_p^1 functions are the associated Legendre functions of the first kind of degree 1 and order p, and the Q_p^1 functions are associated Legendre functions of the second kind.

At each interface the basic laws of magnetostatics reduce to boundary conditions on \bar{B} and \bar{H} (see Equations (30) and (33)) that can be used to determine related boundary conditions on \bar{A} :

$$A_{\psi I} = A_{\psi II} \quad \eta = \eta_1 \quad (48a)$$

$$A_{\psi II} = A_{\psi III} \quad \eta = \eta_2 \quad (48b)$$

$$A_{\psi III} = A_{\psi IV} \quad \eta = \eta_3 \quad (48c)$$

$$\begin{aligned} & -\left(\frac{1}{\mu_1}\right) \frac{1}{a(\xi_1^2 - \nu^2)^{1/2}} \frac{\partial}{\partial \xi} \left[(\xi^2 - 1)^{1/2} A_{\psi II} \right] \Bigg|_{\xi=\xi_1} + \\ & \left(\frac{1}{\mu_1}\right) \frac{1}{a(\xi_1^2 - \nu^2)^{1/2}} \frac{\partial}{\partial \xi} \left[(\xi^2 - 1)^{1/2} A_{\psi I} \right] \Bigg|_{\xi=\xi_1} = \sum_p J_p(\theta) \end{aligned} \quad (48d)$$

$$\left. \left(\frac{1}{\mu_1} \right) \frac{1}{a(\xi_2^2 - \nu^2)^{1/2}} \frac{\partial}{\partial \xi} \left[(\xi^2 - 1)^{1/2} A_{\psi II} \right] \right|_{\xi = \xi_2} =$$

(48e)

$$\left. \left(\frac{1}{\mu_2} \right) \frac{1}{a(\xi_2^2 - \nu^2)^{1/2}} \frac{\partial}{\partial \xi} \left[(\xi^2 - 1)^{1/2} A_{\psi III} \right] \right|_{\xi = \xi_2}$$

$$\left. \left(\frac{1}{\mu_2} \right) \frac{1}{a(\xi_3^2 - \nu^2)^{1/2}} \frac{\partial}{\partial \xi} \left[(\xi^2 - 1)^{1/2} A_{\psi III} \right] \right|_{\xi = \xi_3} =$$

(48f)

$$\left. \left(\frac{1}{\mu_1} \right) \frac{1}{a(\xi_3^2 - \nu^2)^{1/2}} \frac{\partial}{\partial \xi} \left[(\xi^2 - 1)^{1/2} A_{\psi IV} \right] \right|_{\xi = \xi_3}$$

These boundary conditions are then used to evaluate the constants in Equation (47). Using Equations (47) and (48) to solve for the coefficients (where the index p takes on all values from 1 to ∞) we get:

$$H_p P_p^1(\xi_1) P_p^1(\nu) = [I_p P_p^1(\xi_1) + K_p Q_p^1(\xi_1)] P_p^1(\nu) \quad (49a)$$

$$- \left(\frac{1}{\mu_1} \right) \frac{1}{a(\xi_1^2 - \nu^2)^{1/2}} \frac{\partial}{\partial \xi} \left[(\xi^2 - 1)^{1/2} (I_p P_p^1(\xi) + K_p Q_p^1(\xi)) P_p^1(\nu) \right] \Big|_{\xi = \xi_1} +$$

(49b)

$$\left. \left(\frac{1}{\mu_1} \right) \frac{1}{a(\xi_1^2 - \nu^2)^{1/2}} \frac{\partial}{\partial \xi} \left[(\xi^2 - 1)^{1/2} (H_p P_p^1(\xi) P_p^1(\nu)) \right] \right|_{\xi = \xi_1} = J_p(\theta) = \frac{K G P_p^1(\nu)}{a(\xi_1^2 - \nu^2)^{1/2}}$$

$$\left[I_P P_P^1(\xi_2) + K_P Q_P^1(\xi_2) \right] P_P^1(v) = \left[L_P P_P^1(\xi_2) + M_P Q_P^1(\xi_2) \right] P_P^1(v) \quad (49c)$$

$$\left(\frac{1}{\mu_2} \right) \frac{1}{a(\xi_2^2 - v^2)^{1/2}} \frac{\partial}{\partial \xi} \left[(\xi^2 - 1)^{1/2} (L_P P_P^1(\xi) + M_P Q_P^1(\xi)) P_P^1(v) \right] \Bigg|_{\xi=\xi_2} = \quad (49d)$$

$$\left(\frac{1}{\mu_1} \right) \frac{1}{a(\xi_2^2 - v^2)^{1/2}} \frac{\partial}{\partial \xi} \left[(\xi^2 - 1)^{1/2} (I_P P_P^1(\xi) + K_P Q_P^1(\xi)) P_P^1(v) \right] \Bigg|_{\xi=\xi_2}$$

$$\left[L_P P_P^1(\xi_3) + M_P Q_P^1(\xi_3) \right] P_P^1(v) = N_P Q_P^1(\xi_3) P_P^1(v) \quad (49e)$$

$$\left(\frac{1}{\mu_1} \right) \frac{1}{a(\xi_3^2 - v^2)^{1/2}} \frac{\partial}{\partial \xi} \left[(\xi^2 - 1)^{1/2} (N_P Q_P^1(\xi)) P_P^1(v) \right] \Bigg|_{\xi=\xi_3} =$$

$$\left(\frac{1}{\mu_2} \right) \frac{1}{a(\xi_3^2 - v^2)^{1/2}} \frac{\partial}{\partial \xi} \left[(\xi^2 - 1)^{1/2} (L_P P_P^1(\xi) + M_P Q_P^1(\xi)) P_P^1(v) \right] \Bigg|_{\xi=\xi_3}$$

If we make the following substitutions

$$P_P^\Delta(\xi) = \frac{\partial}{\partial \xi} \left[(\xi^2 - 1)^{1/2} P_P^1(\xi) \right] \quad (50)$$

$$Q_P^\Delta(\xi) = \frac{\partial}{\partial \xi} \left[(\xi^2 - 1)^{1/2} Q_P^1(\xi) \right] \quad (51)$$

and perform simple algebraic manipulations, the six boundary conditions reduce to:

$$H_p P_p^1(\xi_1) = I_p P_p^1(\xi_1) + K_p Q_p^1(\xi_1) \quad (52a)$$

$$-\left(\frac{1}{\mu_1}\right)\left(I_p P_p^\Delta(\xi_1) + K_p Q_p^\Delta(\xi_1)\right) + \left(\frac{1}{\mu_1}\right)H_p P_p^\Delta(\xi_1) = \frac{J_p(\theta)a(\xi_1^2 - v^2)^{1/2}}{P_p^1(v)} \quad (52b)$$

$$I_p P_p^1(\xi_2) + K_p Q_p^1(\xi_2) = L_p P_p^1(\xi_2) + M_p Q_p^1(\xi_2) \quad (52c)$$

$$\left(\frac{1}{\mu_2}\right)\left(L_p P_p^\Delta(\xi_2) + M_p Q_p^\Delta(\xi_2)\right) = \left(\frac{1}{\mu_1}\right)\left(I_p P_p^\Delta(\xi_2) + K_p Q_p^\Delta(\xi_2)\right) \quad (52c)$$

$$L_p P_p^1(\xi_3) + M_p Q_p^1(\xi_3) = N_p Q_p^1(\xi_3) \quad (52e)$$

$$\left(\frac{1}{\mu_1}\right)N_p Q_p^\Delta(\xi_3) = \left(\frac{1}{\mu_2}\right)\left(L_p P_p^\Delta(\xi_3) + M_p Q_p^\Delta(\xi_3)\right) \quad (52f)$$

It should be noted in the above equations that the current density $J_\psi(\theta)$ was expanded into a set of associated Legendre functions in order to evaluate the constants in the vector potential components (see Appendix B).

The solution of these six simultaneous equations (52a through 52f) to obtain L_p in terms of known quantities is:

$$L_p = \frac{\left(\frac{1}{\mu_1}\right) \left[J_p^{II} P_p^\Delta(\xi_2) + J_p^I Q_p^\Delta(\xi_2) \right]}{\left(-\frac{1}{\mu_1}\right) [v] P_p^\Delta(\xi_2) + \left(\frac{1}{\mu_2}\right) P_p^\Delta(\xi_2) + \left(\frac{1}{\mu_2}\right) [u] Q_p^\Delta(\xi_2)} \quad (53)$$

where

$$[u] = \frac{\left[\frac{P_p^1(\xi_3)}{Q_p^1(\xi_3)} - \frac{P_p^\Delta(\xi_3)}{Q_p^\Delta(\xi_3)} \left(\frac{\mu_1}{\mu_2} \right) \right]}{\left[\frac{\mu_1}{\mu_2} - 1 \right]} \quad (54)$$

$$[v] = \frac{P_p^1(\xi_2) + [u] Q_p^1(\xi_2)}{P_p^1(\xi_2)} \quad (55)$$

$$J_p^I(\theta) = \frac{K G_P P_p^1(v)}{a(\xi_1^2 - v^2)^{1/2}} \quad (56)$$

$$J_p^I = \frac{\mu_1 J_p^I(\theta) a(\xi_1^2 - v^2)^{1/2}}{P_p^\Delta(\xi_1) P_p^1(v)} \bigg/ \left[\frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} - \frac{Q_p^\Delta(\xi_1)}{P_p^\Delta(\xi_1)} \right] \quad (57)$$

$$J_p^{II} = - J_p^I \frac{Q_p^1(\xi_2)}{P_p^1(\xi_2)} \quad (58)$$

The numerical values for the other five coefficients can be obtained from the following equations:

$$K_p = J_p^I \quad (59)$$

$$M_p = L_p[u] \quad (60)$$

$$I_p = J_p^{II} + L_p[v] \quad (61)$$

$$N_p = L_p \frac{P_p^1(\xi_3)}{Q_p^1(\xi_3)} + M_p \quad (62)$$

$$H_p = I_p + \frac{K_p Q_p^1(\xi_1)}{P_p^1(\xi_1)} \quad (63)$$

The components of the potential A_ψ in regions I-IV can be determined since the coefficients H_p , I_p , K_p , L_p , M_p , and N_p can be calculated for a specific problem. The normal (B_η) and tangential (B_θ) components (to the surface $\eta = \text{constant}$ or $\xi = \text{constant}$) of the magnetic induction in each region I-IV can be determined by using Equations (31) and (35).

APPENDIX A

DERIVATION OF THE VECTOR LAPLACIAN $\nabla^2 \bar{A}_\psi(\eta, \theta)$

We note that a distinction is made between the Laplacian operating on a scalar $\nabla^2 \phi$ and the Laplacian operating on a vector $\nabla^2 \bar{A}(\eta, \theta)$ [i.e., $\nabla^2 \bar{A}(\eta, \theta)$]. The vector Poisson's equation in rectangular coordinates can be treated as three uncoupled scalar equations where $\nabla^2 A_i = J_i$ for $i = x, y, z$. However, if the vector Poisson's equation is resolved into orthogonal components in other coordinate systems the differential operation mixes the components together giving coupled equations.

The vector Laplacian $\nabla^2 \bar{A}$ can be derived by using the well-known vector identity

$$\bar{\nabla} \times (\bar{\nabla} \times \bar{A}) = \bar{\nabla}(\bar{\nabla} \cdot \bar{A}) - \nabla^2 \bar{A} \quad (A-1)$$

In the case herein the Coulomb gage was chosen ($\bar{\nabla} \cdot \bar{A} = 0$); therefore, the following identity applied in our magnetostatic work

$$\nabla^2 \bar{A} = -\bar{\nabla} \times (\bar{\nabla} \times \bar{A}) \quad (A-2)$$

The expression in prolate spheroidal coordinates for $\nabla^2 \bar{A}_\psi(\eta, \theta)$ will now be derived since only the psi component of the vector potential $\bar{A}(\eta, \theta)$ exists in this problem.

Taking the curl of \bar{A} results in the expression

$$\bar{\nabla} \times \bar{A}_\psi(\eta, \theta) =$$

$$\frac{1}{a(\sinh^2 \eta + \sin^2 \theta) \sinh \eta \sin \theta} \times \begin{vmatrix} \bar{e}_\eta (\sinh^2 \eta + \sin^2 \theta)^{1/2} & \bar{e}_\theta (\sinh^2 \eta + \sin^2 \theta)^{1/2} & \bar{e}_\psi \sinh \eta \sin \theta \\ \frac{\partial}{\partial \eta} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \psi} \\ 0 & 0 & A_\psi \sinh \eta \sin \theta \end{vmatrix} \quad (A-3)$$

$$\begin{aligned}
&= \left[\frac{(\sinh^2 \eta + \sin^2 \theta)^{-1/2}}{a \sinh \eta \sin \theta} \right] \times \left[\bar{e}_\eta \frac{\partial}{\partial \theta} (\sinh \eta \sin \theta A_\psi) \right. \\
&\quad \left. - \bar{e}_\theta \frac{\partial}{\partial \eta} (\sinh \eta \sin \theta A_\psi) \right]
\end{aligned}
\tag{A-4}$$

The negative of the curl of $\bar{\nabla} \times \bar{A}(\eta, \theta)$ results in the equation (where $\star^2 \bar{A}_\psi = -\bar{\nabla} \times \bar{\nabla} \times \bar{A}_\psi$)

$$\begin{aligned}
-\bar{\nabla} \times \bar{\nabla} \times \bar{A}_\psi &= \star^2 \bar{A}_\psi = - \left[\frac{1}{a(\sinh^2 \eta + \sin^2 \theta) \sinh \eta \sin \theta} \right] \times \\
&\left| \begin{array}{ccc}
\bar{e}_\eta (\sinh^2 \eta + \sin^2 \theta)^{1/2} & \bar{e}_\theta (\sinh^2 \eta + \sin^2 \theta)^{1/2} & \bar{e}_\psi (\sinh \eta \sin \theta) \\
\frac{\partial}{\partial \eta} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \psi} \\
\frac{1}{a(\sinh \eta \sin \theta)} \frac{\partial}{\partial \theta} (\sinh \eta \sin \theta A_\psi), & \frac{-1}{(\sinh \eta \sin \theta)} \frac{\partial}{\partial \eta} (\sinh \eta \sin \theta A_\psi), & 0
\end{array} \right|
\end{aligned}
\tag{A-5}$$

$$\begin{aligned}
&= \frac{(\sinh^2 \eta + \sin^2 \theta)^{-1/2}}{a(\sinh \eta \sin \theta)} \times \bar{e}_\psi \left[\frac{\partial}{\partial \eta} \frac{1}{a \sinh \eta \sin \theta} \frac{\partial}{\partial \eta} (\sinh \eta \sin \theta A_\psi) \right. \\
&\quad \left. + \frac{\partial}{\partial \theta} \frac{1}{a \sinh \eta \sin \theta} \frac{\partial}{\partial \theta} (\sinh \eta \sin \theta A_\psi) \right]
\end{aligned}
\tag{A-6}$$

$$= \frac{(\sinh^2 \eta + \sin^2 \theta)^{-1/2}}{a^2 (\sinh \eta \sin \theta)} \bar{e}_\psi \left(\frac{\partial}{\partial \eta} \frac{1}{\sinh \eta} \frac{\partial}{\partial \eta} \sinh \eta A_\psi \right)$$

(A-7)

$$+ \frac{\partial}{\partial \theta} \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_\psi)$$

Now, if \bar{A}_ψ is set equal to zero, then the vector potential $A_\psi(\eta, \theta)$ will satisfy the equation (see Equation (19) in text of report).

$$\frac{\partial}{\partial \eta} \left[\frac{1}{\sinh \eta} \frac{\partial}{\partial \eta} (\sinh \eta A_\psi) \right] + \frac{\partial}{\partial \theta} \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_\psi) \right] = 0 \quad (A-8)$$

It is also interesting to note at this point that $\bar{\nabla} \cdot \bar{A}_\psi(\eta, \theta) = 0$ as it must, since the Coulomb gage was chosen. The divergence of the vector \bar{A} in prolate spheroidal coordinates is

$$\bar{\nabla} \cdot \bar{A} =$$

$$\frac{1}{a(\sinh^2 \eta + \sin^2 \theta)} \left\{ \frac{1}{\sinh \eta} \frac{\partial}{\partial \eta} \left[(\sinh^2 \eta + \sin^2 \theta)^{1/2} \sinh \eta A_\eta \right] \right. \quad (A-9)$$

$$\left. + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left[(\sinh^2 \eta + \sin^2 \theta)^{1/2} \sin \theta A_\theta \right] \right\} + \frac{1}{a \sinh \eta \sin \theta} \frac{\partial A_\psi}{\partial \psi}$$

Since the two components A_η and A_θ are zero in our case, and A_ψ is a function of η and θ only, then $\bar{\nabla} \cdot \bar{A}_\psi(\eta, \theta)$ is equal to zero as required.

APPENDIX B
EXPANSION OF CURRENT $J_\psi(\theta)$ IN ASSOCIATED
LEGENDRE FUNCTIONS $P_p^1(\cos\theta)$

Let us assume the expansion is similar to Purczynski's⁴ (see Equation (56)).

$$J_\psi(\theta) = \frac{J \sum_{p=1}^{\infty} G_p P_p^1(\cos\theta)}{a(\sinh^2 \eta_1 + \sin^2 \theta)^{1/2}} \quad (B-1)$$

In this case the current band was chosen to be on boundary $\eta = \eta_1$.

The coefficients G_p are determined from Equation (B-1) by multiplying both sides by $\sin\theta P_{p'}^1(\cos\theta)$ and $a(\sinh^2 \eta_1 + \sin^2 \theta)^{1/2}$ and integrating

$$\int_0^\pi J_\psi(\theta) a(\sinh^2 \eta_1 + \sin^2 \theta)^{1/2} P_{p'}^1(\cos\theta) \sin\theta d\theta =$$

$$\int_0^\pi J G_p P_p^1(\cos\theta) P_{p'}^1(\cos\theta) \frac{a(\sinh^2 \eta_1 + \sin^2 \theta)^{1/2}}{a(\sinh^2 \eta_1 + \sin^2 \theta)^{1/2}} \sin\theta d\theta = \quad (B-2)$$

$$J G_p \frac{2p(p+1)}{2p+1}$$

where the identity

$$\int_0^\pi P_{p'}^m(\cos\theta) P_p^m(\cos\theta) \sin\theta d\theta = \frac{2(p+m)! \delta_{p,p'}}{(2p+1)(p-m)!}$$

was used to simplify Equation (B-2).

When $J_\psi(\theta)$ is set conveniently to a constant J , G_p can be expressed as

$$G_p = \frac{(2p+1)a}{2p(p+1)} \int_0^\pi \left(\sinh^2 \eta_1 + \sin^2 \theta \right)^{1/2} P_p^1(\cos \theta) \sin \theta d\theta \quad (B-3)$$

Since the current in this problem extends from θ_1 to θ_2 the expression for G_p can be further modified to read

$$G_p = \frac{(2p+1)a}{2p(p+1)} \int_{\theta_1}^{\theta_2} \left(\sinh^2 \eta_1 + \sin^2 \theta \right)^{1/2} P_p^1(\cos \theta) \sin \theta d\theta \quad (B-4)$$

By making the following substitutions

$$\xi = \cosh \eta, \quad v = \cos \theta \quad (B-5)$$

G_p can be conveniently written as

$$G_p = - \frac{(2p+1)a}{2p(p+1)} \int_{v_1}^{v_2} \left(\xi_1^2 - v^2 \right)^{1/2} P_p^1(v) dv \quad (B-6)$$

This integral can be evaluated with much difficulty in closed mathematical form. Purczynski⁴ evaluated these integrals for C_{2p} (even terms) and C_{2p+1} (odd terms) in terms of very complicated series expansions.

In the computer work to be performed, the integrals for G_p will be evaluated numerically, the reason being it has been found that a large number of coefficients were necessary to obtain a good fit for the current expanded in terms of associated Legendre functions for the analogous spherical shell problem.

APPENDIX C
DERIVATION OF THE COEFFICIENTS FOR THE VECTOR.
POTENTIALS FOR A THIN COIL SURROUNDING
A FERROMAGNETIC SPHEROIDAL SHELL

In this appendix the coefficients are derived for the vector potential in regions I-IV for a ferromagnetic spheroidal shell surrounded by an infinitely thin current band. For a detailed discussion of the ferromagnetic problem see the section in the text of this report entitled "Thin Coil Surrounding a Ferromagnetic Prolate Spheroidal Shell." The magnetic vector potentials in each region are given by:

$$A_{\psi I} = \sum_{p=1}^{\infty} \left[A_p P_p^1(\xi) \right] P_p^1(\nu) \quad (C-1a)$$

$$A_{\psi II} = \sum_{p=1}^{\infty} \left[B_p P_p^1(\xi) + C_p Q_p^1(\xi) \right] P_p^1(\nu) \quad (C-1b)$$

$$A_{\psi III} = \sum_{p=1}^{\infty} \left[D_p P_p^1(\xi) + E_p Q_p^1(\xi) \right] P_p^1(\nu) \quad (C-1c)$$

$$A_{\psi IV} = \sum_{p=1}^{\infty} \left[F_p Q_p^1(\xi) \right] P_p^1(\nu) \quad (C-1d)$$

The coefficients in Equations (C-1a) to (C-1d) are obtained by substituting these equations into Equations (C-2a) to C-2f).

$$A_{\psi I} = A_{\psi II} \quad \text{at } \eta = \eta_1 \quad (C-2a)$$

$$A_{\psi II} = A_{\psi III} \quad \text{at } \eta = \eta_2 \quad (C-2b)$$

$$A_{\psi III} = A_{\psi IV} \quad \text{at } \eta = \eta_3 \quad (C-2c)$$

$$\left. \left(\frac{1}{\mu_2} \right) \frac{1}{a(\xi_1^2 - v^2)^{1/2}} \frac{\partial}{\partial \xi} \left[(\xi^2 - 1)^{1/2} A_{\psi II} \right] \right|_{\xi=\xi_1} = \quad (C-2d)$$

$$\left. \left(\frac{1}{\mu_1} \right) \frac{1}{a(\xi_1^2 - v^2)^{1/2}} \frac{\partial}{\partial \xi} \left[(\xi^2 - 1)^{1/2} A_{\psi I} \right] \right|_{\xi=\xi_1}$$

$$\left. \left(\frac{1}{\mu_1} \right) \frac{1}{a(\xi_2^2 - v^2)^{1/2}} \frac{\partial}{\partial \xi} \left[(\xi^2 - 1)^{1/2} A_{\psi III} \right] \right|_{\xi=\xi_2} = \quad (C-2e)$$

$$\left. \left(\frac{1}{\mu_2} \right) \frac{1}{a(\xi_2^2 - v^2)^{1/2}} \frac{\partial}{\partial \xi} \left[(\xi^2 - 1)^{1/2} A_{\psi II} \right] \right|_{\xi=\xi_2}$$

$$\left. - \left(\frac{1}{\mu_1} \right) \frac{1}{a(\xi_3^2 - v^2)^{1/2}} \frac{\partial}{\partial \xi} \left[(\xi^2 - 1)^{1/2} A_{\psi IV} \right] \right|_{\xi=\xi_3}$$

$$+ \left. \left(\frac{1}{\mu_1} \right) \frac{1}{a(\xi_3^2 - v^2)^{1/2}} \frac{\partial}{\partial \xi} \left[(\xi^2 - 1)^{1/2} A_{\psi III} \right] \right|_{\xi=\xi_3} \quad (C-2f)$$

$$= \sum_P J_P = \sum_P \frac{K_{PP} G_{PP}^1(v)}{a(\xi_3^2 - v^2)^{1/2}}$$

Let us define the following functions as

$$P_p^\Delta(\xi) = \frac{d}{d\xi} \left[(\xi^2 - 1)^{1/2} P_p^1(\xi) \right] \quad (C-2g)$$

$$Q_p^\Delta(\xi) = \frac{d}{d\xi} \left[(\xi^2 - 1)^{1/2} Q_p^1(\xi) \right] \quad (C-2h)$$

After appropriate substitution of Equations (C-1a) to (C-1d) into Equations (C-2a) to (C-2f) and using (C-2g) and (C-2h), the following boundary value equations are obtained.

$$A_p P_p^1(\xi_1) = B_p P_p^1(\xi_1) + C_p Q_p^1(\xi_1) \quad (C-3a)$$

$$\left(\frac{1}{\mu_2} \right) \left[B_p P_p^\Delta(\xi_1) + C_p Q_p^\Delta(\xi_1) \right] = \left(\frac{1}{\mu_1} \right) A_p P_p^\Delta(\xi_1) \quad (C-3b)$$

$$B_p P_p^1(\xi_2) + C_p Q_p^1(\xi_2) = D_p P_p^1(\xi_2) + E_p Q_p^1(\xi_2) \quad (C-3c)$$

$$\left(\frac{1}{\mu_1} \right) \left[D_p P_p^\Delta(\xi_2) + E_p Q_p^\Delta(\xi_2) \right] = \left(\frac{1}{\mu_2} \right) \left[B_p P_p^\Delta(\xi_2) + C_p Q_p^\Delta(\xi_2) \right] \quad (C-3d)$$

$$D_p P_p^1(\xi_3) + E_p Q_p^1(\xi_3) = F_p Q_p^1(\xi_3) \quad (C-3e)$$

$$-\left(\frac{1}{\mu_1}\right)F_P Q_P^\Delta(\xi_3) + \left(\frac{1}{\mu_1}\right)D_P P_P^\Delta(\xi_3) + \left(\frac{1}{\mu_1}\right)E_P Q_P^\Delta(\xi_3) = \frac{J_P a(\xi_3^2 - v^2)^{1/2}}{P_P^1(v)} \quad (C-3f)$$

where

$$J_P = \frac{k G P_P^1(v)}{a(\xi^2 - v^2)^{1/2}}$$

These algebraic equations provide six simultaneous equations with six unknowns, and they can be solved for the coefficients A_P , B_P , C_P , D_P , E_P , and F_P .

After solving Equation (C-3a) for A_P , we obtain

$$A_P = B_P + C_P \frac{Q_P^1(\xi_1)}{P_P^1(\xi_1)} \quad (C-4)$$

The solution of A_P from Equation (C-3b) is

$$A_P = B_P \left(\frac{\mu_1}{\mu_2} \right) + C_P \frac{\mu_1}{\mu_2} \left(\frac{Q_P^\Delta(\xi_1)}{P_P^\Delta(\xi_1)} \right) \quad (C-5)$$

By equating Equations (C-4) and (C-5) and solving for B_P , one obtains

$$B_P = \frac{C_P \left[\frac{\mu_1}{\mu_2} \left(\frac{Q_P^\Delta(\xi_1)}{P_P^\Delta(\xi_1)} \right) - \frac{Q_P^1(\xi_1)}{P_P^1(\xi_1)} \right]}{\left[1 - \frac{\mu_1}{\mu_2} \right]}$$

where

$$B_p = [x] C_p \quad (C-7)$$

and

$$[x] = \frac{\left[\frac{\mu_1}{\mu_2} \left(\frac{Q_p^\Delta(\xi_1)}{P_p^\Delta(\xi_1)} \right) - \frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} \right]}{\left[1 - \frac{\mu_1}{\mu_2} \right]}$$

Solving Equation (C-3e) for F_p , we derive the result

$$F_p = D_p \frac{P_p^1(\xi_3)}{Q_p^1(\xi_3)} + E_p \quad (C-8)$$

In addition, the solution for F_p from Equation (C-3f) is

$$F_p = \frac{-\mu_1 J_p a (\xi_3^2 - v^2)^{1/2}}{Q_p^\Delta(\xi_3) P_p^1(v)} + \frac{D_p P_p^\Delta(\xi_3)}{Q_p^\Delta(\xi_3)} + E_p \quad (C-9)$$

Equating (C-8) and (C-9) and solving for D_p , one derives the result

$$D_p = J_p^I \quad (C-10)$$

where

$$J_p^I = \left[\frac{-\mu_1 J_p a (\xi_3^2 - \nu^2)^{1/2}}{P_p^1(\nu) Q_p^\Delta(\xi_3)} \right] / \left[\frac{P_p^1(\xi_3)}{Q_p^1(\xi_3)} - \frac{P_p^\Delta(\xi_3)}{Q_p^\Delta(\xi_3)} \right]$$

Solving Equation (C-3c) for C_p and using Equations (C-7) and (C-10), we arrive at the result

$$C_p = \frac{J_p^I P_p^1(\xi_2)}{([x] P_p^1(\xi_2) + Q_p^1(\xi_2))} + \frac{E_p Q_p^1(\xi_2)}{([x] P_p^1(\xi_2) + Q_p^1(\xi_2))} \quad (C-11)$$

This can be expressed as

$$C_p = J_p^{II} + E_p [z] \quad (C-12)$$

where

$$J_p^{II} = \frac{J_p^I P_p^1(\xi_2)}{([x] P_p^1(\xi_2) + Q_p^1(\xi_2))}$$

$$[z] = \frac{Q_p^1(\xi_2)}{([x] P_p^1(\xi_2) + Q_p^1(\xi_2))}$$

Using Equation (C-3d) to solve for E_p and substituting Equations (C-7), (C-10), and (C-12), we arrive at the expression

$$E_p = \frac{-\frac{1}{\mu_2} [x] J_p^{II} P_p^\Delta(\xi_2) - \frac{1}{\mu_2} \left(J_p^{II} \right) Q_p^\Delta(\xi_2) + \frac{1}{\mu_1} J_p^I P_p^\Delta(\xi_2)}{\frac{1}{\mu_2} \left([x] [z] P_p^\Delta(\xi_2) \right) + \frac{1}{\mu_2} \left([z] Q_p^\Delta(\xi_2) \right) - \frac{1}{\mu_1} Q_p^\Delta(\xi_2)} \quad (C-13)$$

The constants are thus determined by using Equations (C-4), (C-7), (C-8), (C-10), (C-12), and (C-13). For convenience, (C-4), (C-7), (C-8), (C-10), and (C-12) are listed below

$$C_p = J_p^{II} + E_p [z]$$

$$B_p = [x] C_p$$

$$D_p = J_p^I$$

(C-14)

$$A_p = B_p + C_p \frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)}$$

$$F_p = D_p \frac{P_p^1(\xi_3)}{Q_p^1(\xi_2)} + E_p$$

APPENDIX D
DERIVATION OF THE COEFFICIENTS FOR THE VECTOR
POTENTIAL FOR A THIN COIL INTERNAL TO
A FERROMAGNETIC SPHEROIDAL SHELL

In this appendix the coefficients are derived for the vector potentials in regions I-IV for an infinitely thin current band surrounded by a ferromagnetic spheroidal shell. For a detailed discussion of the ferromagnetic problem see the section in the text of this report entitled "Thin Coil Internal to a Ferromagnetic Prolate Spheroidal Shell." The magnetic vector potential in each region is given by:

$$A_{\psi I} = \sum_{p=1}^{\infty} \left[H_p P_p^1(\xi) \right] P_p^1(\nu) \quad (D-1a)$$

$$A_{\psi II} = \sum_{p=1}^{\infty} \left[I_p P_p^1(\xi) + K_p Q_p^1(\xi) \right] P_p^1(\nu) \quad (D-1b)$$

$$A_{\psi III} = \sum_{p=1}^{\infty} \left[L_p P_p^1(\xi) + M_p Q_p^1(\xi) \right] P_p^1(\nu) \quad (D-1c)$$

$$A_{\psi IV} = \sum_{p=1}^{\infty} \left[N_p Q_p^1(\xi) \right] P_p^1(\nu) \quad (D-1d)$$

The coefficients in Equations (D-1a) to (D-1d) are obtained by substituting these equations into the boundary conditions (Equations (D-2a) to (D-2f)).

$$A_{\psi I} = A_{\psi II} \quad \text{at } \eta = \eta_1 \quad (D-2a)$$

$$A_{\psi II} = A_{\psi III} \quad \text{at } \eta = \eta_2 \quad (D-2b)$$

$$A_{\psi III} = A_{\psi IV} \quad \text{at } \eta = \eta_3 \quad (D-2c)$$

$$-\left(\frac{1}{\mu_1}\right) \frac{1}{a(\xi_1^2 - v^2)^{1/2}} \frac{\partial}{\partial \xi} \left[(\xi^2 - 1)^{1/2} A_{\psi \text{II}} \right] \Bigg|_{\xi=\xi_1} + \quad (\text{D-2d})$$

$$\left(\frac{1}{\mu_1}\right) \frac{1}{a(\xi_1^2 - v^2)^{1/2}} \frac{\partial}{\partial \xi} \left[(\xi^2 - 1)^{1/2} A_{\psi \text{I}} \right] \Bigg|_{\xi=\xi_1} = \sum_p J_p(\theta)$$

$$\left(\frac{1}{\mu_1}\right) \frac{1}{a(\xi_2^2 - v^2)^{1/2}} \frac{\partial}{\partial \xi} \left[(\xi^2 - 1)^{1/2} A_{\psi \text{II}} \right] \Bigg|_{\xi=\xi_2} = \quad (\text{D-2e})$$

$$\left(\frac{1}{\mu_2}\right) \frac{1}{a(\xi_2^2 - v^2)^{1/2}} \frac{\partial}{\partial \xi} \left[(\xi^2 - 1)^{1/2} A_{\psi \text{III}} \right] \Bigg|_{\xi=\xi_2}$$

$$\left(\frac{1}{\mu_2}\right) \frac{1}{a(\xi_3^2 - v^2)^{1/2}} \frac{\partial}{\partial \xi} \left[(\xi^2 - 1)^{1/2} A_{\psi \text{III}} \right] \Bigg|_{\xi=\xi_3} = \quad (\text{D-2f})$$

$$\left(\frac{1}{\mu_1}\right) \frac{1}{a(\xi_3^2 - v^2)^{1/2}} \frac{\partial}{\partial \xi} \left[(\xi^2 - 1)^{1/2} A_{\psi \text{IV}} \right] \Bigg|_{\xi=\xi_3}$$

Let us define the following functions as

$$P_p^\Delta(\xi) = \frac{d}{d\xi} \left[(\xi^2 - 1)^{1/2} P_p^1(\xi) \right]$$

$$Q_p^\Delta(\xi) = \frac{d}{d\xi} \left[(\xi^2 - 1)^{1/2} Q_p^1(\xi) \right]$$

After appropriate substitution of Equations (D-1a) to (D-1d) into Equations (D-2a) to (D-2f), the following boundary value equations are obtained.

$$H_p P_p^1(\xi_1) = I_p P_p^1(\xi_1) + K_p Q_p^1(\xi_1) \quad (D-3a)$$

$$-\left(\frac{1}{\mu_1}\right) \left(I_p P_p^\Delta(\xi_1) + K_p Q_p^\Delta(\xi_1) \right) + \left(\frac{1}{\mu_1}\right) H_p P_p^\Delta(\xi_1) = \frac{J_p a(\xi_1^2 - v^2)^{1/2}}{P_p^1(v)} \quad (D-3b)$$

$$I_p P_p^1(\xi_2) + K_p Q_p^1(\xi_2) = L_p P_p^1(\xi_2) + M_p Q_p^1(\xi_2) \quad (D-3c)$$

$$\left(\frac{1}{\mu_2}\right) \left(L_p P_p^\Delta(\xi_2) + M_p Q_p^\Delta(\xi_2) \right) = \left(\frac{1}{\mu_1}\right) \left(I_p P_p^\Delta(\xi_2) + K_p Q_p^\Delta(\xi_2) \right) \quad (D-3d)$$

$$L_p P_p^1(\xi_3) + M_p Q_p^1(\xi_3) = N_p Q_p^1(\xi_3) \quad (D-3e)$$

$$\left(\frac{1}{\mu_1}\right) N_p Q_p^\Delta(\xi_3) = \left(\frac{1}{\mu_2}\right) \left(L_p P_p^\Delta(\xi_3) + M_p Q_p^\Delta(\xi_3) \right) \quad (D-3f)$$

These algebraic equations provide six simultaneous equations with six unknowns, and they can be solved for the coefficients H_p , I_p , K_p , M_p , and N_p .

Solving Equation (D-3a) for H_p in terms of I_p and K_p , we arrive at the equation

$$H_p = I_p + K_p \frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} \quad (D-4)$$

Solving Equation (D-3b) for H_p , we obtain the equation

$$H_p = I_p + K_p \frac{Q_p^\Delta(\xi_1)}{P_p^\Delta(\xi_1)} + \frac{\mu_1 J_p a (\xi_1^2 - v^2)^{1/2}}{P_p^\Delta(\xi_1) P_p^1(v)} \quad (D-5)$$

Equating Equations (D-4) and (D-5) and solving for K_p yields

$$K_p = J_p^I \quad (D-6)$$

where

$$J_p^I = \left[\frac{\mu_1 J_p a (\xi_1^2 - v^2)^{1/2}}{P_p^\Delta(\xi_1) P_p^1(v)} \right] / \left[\frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} - \frac{Q_p^\Delta(\xi_1)}{P_p^\Delta(\xi_1)} \right]$$

Now solving Equation (D-3e) for N_p , we obtain

$$N_p = L_p \frac{P_p^1(\xi_3)}{Q_p^1(\xi_3)} + M_p \quad (D-7)$$

Additionally, by solving Equation (D-3f) for N_p , one obtains the expression

$$N_p = \frac{\left(\frac{\mu_1}{\mu_2} \right) \left(L_p P_p^\Delta(\xi_3) + M_p Q_p^\Delta(\xi_3) \right)}{Q_p^\Delta(\xi_3)} \quad (D-8)$$

By equating Equations (D-7) and (D-8), we may obtain M_p in terms of L_p

$$M_p = L_p [u] \quad (D-9)$$

where

$$[u] = \frac{\left[\frac{P_p^1(\xi_3)}{Q_p^1(\xi_3)} - \frac{P_p^\Delta(\xi_3)}{Q_p^\Delta(\xi_3)} \left(\frac{\mu_1}{\mu_2} \right) \right]}{\left[\frac{\mu_1}{\mu_2} - 1 \right]}$$

Solving (D-3c) for I_p and using Equations (D-6) and (D-9), we derive the result

$$I_p = J_p^{II} + L_p [v] \quad (D-10)$$

where

$$J_p^{II} = - J_p^I \frac{Q_p^1(\xi_2)}{P_p^1(\xi_2)}$$

$$[v] = \frac{P_p^1(\xi_2) + [u] Q_p^1(\xi_2)}{P_p^1(\xi_2)}$$

Now if we use Equation (D-3d) and substitute from Equations (D-6), (D-9), and (D-10) the constant L_p can be determined to be

$$L_p = \frac{\left(\frac{1}{\mu_1} \right) \left[J_p^{II} P_p^\Delta(\xi_2) + J_p^I Q_p^\Delta(\xi_2) \right]}{-\left(\frac{1}{\mu_1} \right) [v] P_p^\Delta(\xi_2) + \left(\frac{1}{\mu_2} \right) P_p^\Delta(\xi_2) + \left(\frac{1}{\mu_2} \right) [u] Q_p^\Delta(\xi_2)} \quad (D-11)$$

The constants have now been found. After the numerical value for L_p is calculated on the computer for a specific problem, the numerical values for the other coefficients can be obtained from the following equations:

$$K_p = J_p^I$$

$$M_p = L_p [u]$$

$$I_p = J_p^{II} + L_p [v] \quad (D-12)$$

$$N_p = L_p \frac{P_p^1(\xi_3)}{Q_p^1(\xi_3)} + M_p$$

$$H_p = I_p + K_p \frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)}$$

APPENDIX E

DETERMINATION OF THE MAGNETIC VECTOR POTENTIAL FOR AN INFINITESIMALLY THIN PROLATE SPHEROIDAL CURRENT BAND

In this appendix the Potentials $A_{\psi I}$ in the inner region and $A_{\psi II}$ in outer region are derived for the infinitesimally thin current band in a homogeneous medium of permeability μ_1 (see Figure E-1).

The potential in the inner region $A_{\psi I}$ and the outer region $A_{\psi II}$ of the infinitesimally thin current band problem are solutions to the vector Laplace's equation $\nabla \cdot \bar{A} = 0$. These solutions can be expressed as:

$$A_{\psi I} = \sum_{p=1}^{\infty} A_p P_p^1(\xi) P_p^1(\nu) \quad (E-1a)$$

$$A_{\psi II} = \sum_{p=1}^{\infty} F_p Q_p^1(\xi) P_p^1(\nu) \quad (E-1b)$$

The coefficients A_p and F_p are determined from the boundary conditions of the problem. After algebraic manipulation such as with Equations (32) and (36) in the text, the boundary conditions for the normal component of \bar{B} and the tangential component of \bar{H} becomes:

$$A_{\psi I} = A_{\psi II} \text{ at } \eta = \eta_1 \quad (E-2a)$$

$$\begin{aligned} & -\left(\frac{1}{\mu_1}\right) \frac{1}{a(\xi_1^2 - \nu^2)^{1/2}} \frac{\partial}{\partial \xi} \left[(\xi^2 - 1)^{1/2} A_{\psi II} \right] \Big|_{\xi=\xi_1} + \\ & \left(\frac{1}{\mu_1}\right) \frac{1}{a(\xi_1^2 - \nu^2)^{1/2}} \frac{\partial}{\partial \xi} \left[(\xi^2 - 1)^{1/2} A_{\psi I} \right] \Big|_{\xi=\xi_1} = \sum_p J_p(\theta) \end{aligned} \quad (E-2b)$$

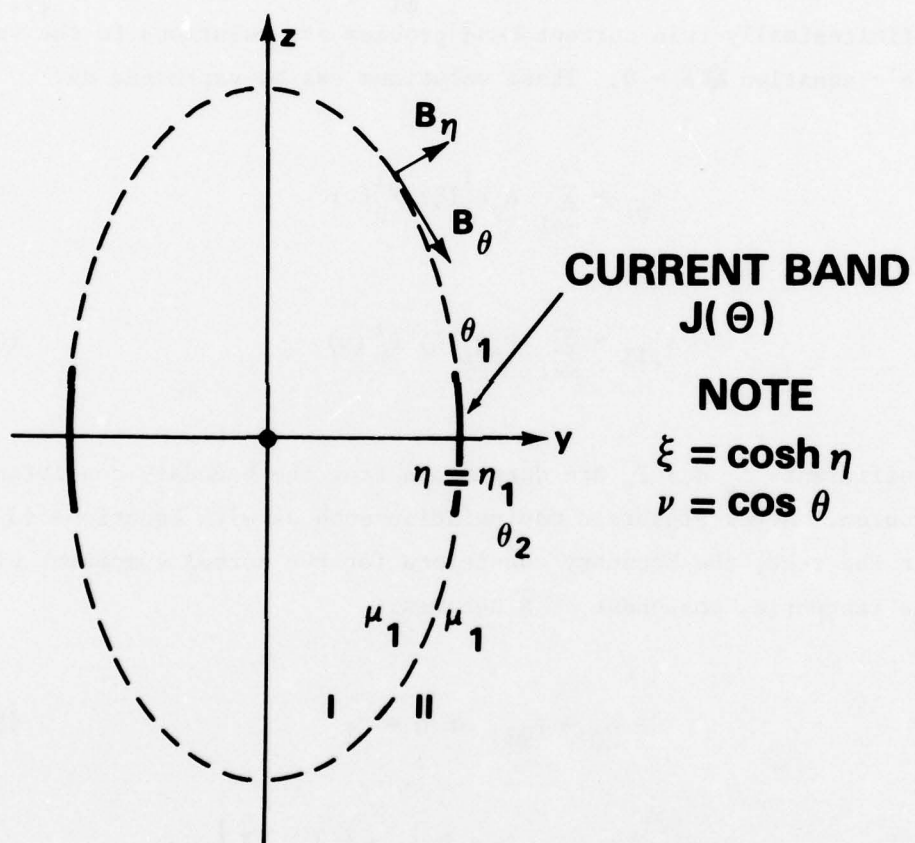


Figure E-1 - Infinitesimally Thin Spheroidal Current Band

Using the relationship $\xi = \cosh \eta$ and $v = \cos \theta$ and substituting the expressions for $A_{\psi I}$ and $A_{\psi II}$ (Equations (E-1a) and (E-1b)) into the boundary value equations (Equations (E-2a) and (E-2b)) provides us with the following algebraic equations for the coefficients:

$$A_p P_p^1(\xi_1) = F_p Q_p^1(\xi_1) \quad (E-3a)$$

$$-\frac{1}{\mu_1} \left[F_p Q_p^\Delta(\xi_1) \right] + \frac{1}{\mu_1} \left[A_p P_p^\Delta(\xi_1) \right] = \frac{J_p(\theta) a(\xi_1^2 - v^2)^{1/2}}{P_p^1(v)}$$

The following substitutions were used to simplify the above expressions.

$$P_p^\Delta(\xi) = \frac{d}{d\xi} \left[(\xi^2 - 1)^{1/2} P_p^1(\xi) \right] \quad (E-4)$$

$$Q_p^\Delta(\xi) = \frac{d}{d\xi} \left[(\xi^2 - 1)^{1/2} Q_p^1(\xi) \right] \quad (E-5)$$

These equations are solved for A_p and F_p by simple algebraic manipulation.

$$A_p = F_p \frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} \quad (E-6a)$$

$$F_p = \left[\frac{\mu_1 J_p(\theta) a(\xi_1^2 - v^2)^{1/2}}{P_p^\Delta(\xi_1) P_p^1(v)} \right] / \left[\frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} - \frac{Q_p^\Delta(\xi_1)}{P_p^\Delta(\xi_1)} \right] \quad (E-6b)$$

The potential $A_{\psi I}$ and $A_{\psi II}$ are determined by substituting the expression for A_p and F_p into Equations (E-1a) and (E-1b)

$$A_{\psi I} = \sum_{p=1}^{\infty} \left\{ \left[\frac{\mu_1 J_p(\theta) a (\xi_1^2 - v^2)^{1/2}}{P_p^{\Delta}(\xi_1) P_p^1(v)} - \left(\frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} \right) \right] / \left[\frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} - \frac{Q_p^{\Delta}(\xi_1)}{P_p^{\Delta}(\xi_1)} \right] \right\} P_p^1(\xi) P_p^1(v) \quad (E-7a)$$

$$A_{\psi II} = \sum_{p=1}^{\infty} \left\{ \frac{\mu_1 J_p(\theta) a (\xi_1^2 - v^2)^{1/2}}{P_p^{\Delta}(\xi_1) P_p^1(v)} / \left[\frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} - \frac{Q_p^{\Delta}(\xi_1)}{P_p^{\Delta}(\xi_1)} \right] \right\} Q_p^1(\xi) P_p^1(v) \quad (E-7b)$$

where

$$J_p(\theta) = \frac{K G P_p^1(v)}{a (\xi_1^2 - v^2)^{1/2}} \quad (E-8)$$

APPENDIX F

REDUCTION OF THE MAGNETIC VECTOR POTENTIAL FOR A THIN COIL SURROUNDING A FERROMAGNETIC SPHEROIDAL SHELL TO THAT OF A THIN COIL IN FREE SPACE WHEN IN THE LIMIT μ_2 EQUALS μ_1

In this appendix, the coefficients A_p , B_p , C_p , D_p , E_p , and F_p for the potentials are evaluated for the system consisting of a ferromagnetic shell with permeability μ_2 surrounded by an infinitesimally thin current band in a homogeneous medium with permeability μ_1 in the limit as $\mu_2 = \mu_1$. These coefficients are utilized in Equation (29) in the section of this report entitled "Thin Coil Surrounding a Ferromagnetic Prolate Spheroidal Shell." The variables are defined in Figure 2 located in the text. When μ_2 is set equal to μ_1 the problem reduces to that of finding the potentials in the two regions of a simple current band (see Figure 1-E in Appendix E), since the ferromagnetic shell will now have a permeability μ_1 equal to that of the homogeneous medium with permeability μ_1 .

In this limit the coefficients should assume the following form:

$$A_p = B_p = D_p \quad (F-1a)$$

$$C_p = E_p = 0 \quad (F-1b)$$

and where A_p and F_p should reduce to the coefficients for the potentials in the two regions for the spheroidal band problem (see Appendix E). If the coefficients assume this mathematical form it will prove that the mathematical form of the coefficients for the spheroidal shell surrounded by a thin current band are mathematically correct.

The mathematical solution for E_p in terms of known quantities was derived in Appendix C and was reported in the text of this report (see Equation (41a)).

$$E_p = \frac{-\frac{1}{\mu_2}([x]J_p^{II}P_p^\Delta(\xi_2)) - \frac{1}{\mu_2}J_p^{II}Q_p^\Delta(\xi_2) + \frac{1}{\mu_1}(J_p^I)P_p^\Delta(\xi_2)}{\frac{1}{\mu_2}([x][z]P_p^\Delta(\xi_2)) + \frac{1}{\mu_2}([z]Q_p^\Delta(\xi_2)) - \left(\frac{1}{\mu_1}\right)Q_p^\Delta(\xi_2)} \quad (F-2a)$$

$$[x] = \frac{\left(\frac{\mu_1}{\mu_2}\right)\frac{Q_p^\Delta(\xi_1)}{P_p^\Delta(\xi_1)} - \frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)}}{\left(1 - \frac{\mu_1}{\mu_2}\right)} \quad (F-2b)$$

$$[z] = \frac{Q_p^1(\xi_2)}{[x]P_p^1(\xi_2) + Q_p^1(\xi_2)} \quad (F-2c)$$

$$J_p^I = \left\{ \frac{-\mu_1 J_p(\theta) a(\xi_3^2 - v^2)^{1/2}}{P_p^1(v) Q_p^\Delta(\xi_3)} \right\} / \left[\frac{P_p^1(\xi_3)}{Q_p^1(\xi_3)} - \frac{P_p^\Delta(\xi_3)}{Q_p^\Delta(\xi_3)} \right] \quad (F-2d)$$

$$J_p^{II} = \frac{J_p^I P_p^1(\xi_2)}{[x]P_p^1(\xi_2) + Q_p^1(\xi_2)} \quad (F-2e)$$

The coefficient E_p will now be evaluated when the limit is taken with $\mu_2 = \mu_1$ which cause $[x]$ to approach infinity (∞). Also the expression for J_p^{II} is substituted into Equation (F-2a).

$$E_p \Big|_{\mu_2 = \mu_1} = \lim_{[x] \rightarrow \infty} E_p$$

$$E_p \Big|_{\mu_2 = \mu_1} = \lim_{[x] \rightarrow \infty} \frac{-\frac{1}{\mu_1} [x] \left\{ \frac{J_p^1(\xi)}{[x] p_p^1(\xi_2) + q_p^1(\xi_2)} - \frac{1}{\mu_1} \right\} \frac{J_p^1(\xi_2)}{[x] p_p^1(\xi_2) + q_p^1(\xi_2)} \left\{ p_p^\Delta(\xi_2) - \frac{1}{\mu_1} \right\} + \frac{1}{\mu_1} (J_p^1) p_p^\Delta(\xi_2)}{\frac{1}{\mu_1} [x] \left(\frac{q_p^1(\xi_2)}{[x] p_p^1(\xi_2) + q_p^1(\xi_2)} \right) p_p^\Delta(\xi_2) + \frac{1}{\mu_1} \left(\frac{q_p^1(\xi_2) q_p^\Delta(\xi_2)}{[x] p_p^1(\xi_2) + q_p^1(\xi_2)} \right) - \frac{1}{\mu_1} q_p^\Delta(\xi_2)}$$

$$= \frac{-\frac{1}{\mu_1} J_{pp}^{\Delta}(\varepsilon_2) + \frac{1}{\mu_1} J_{pp}^{\Delta}(\varepsilon_2)}{\frac{1}{\mu_1} \left[\frac{Q^1(\varepsilon_2) p^{\Delta}(\varepsilon_2)}{p^1(\varepsilon_2)} - p^1(\varepsilon_2) Q_p^{\Delta}(\varepsilon_2) \right]}$$

$$\frac{E_p}{\mu_2 = \mu_1} = 0$$

(F-3)

The expression for B_p (see Equations (42b) and (41b)) is

$$B_p = [x] C_p = [x] J_p^{II} + E_p [x] [z] \quad (F-4a)$$

where

$$C_p = J_p^{II} + E_p [z] \quad (F-4b)$$

The expression for B_p when μ_2 equals μ_1 can be expressed as

$$B_p \Big|_{\mu_2=\mu_1} = \lim_{\mu_2 \rightarrow \mu_1} \left\{ [x] \left[E_p \Big|_{\mu_2=\mu_1} \right] [z] \right\} + \lim_{\mu_2 \rightarrow \mu_2} \left([x] J_p^{II} \right) \quad (F-5)$$

$$B_p \Big|_{\mu_2 \rightarrow \mu_1} = J_p^I$$

where

$$\lim_{\mu_2 \rightarrow \mu_1} [x] [z] = Q_p^1(\xi_2) / P_p^1(\xi_2)$$

and

$$\lim_{\mu_2 \rightarrow \mu_1} [x] J_p^{II} = \lim_{[x] \rightarrow \infty} \left[\frac{[x] J_p^I P_p^1(\xi_2)}{[x] P_p^1(\xi_2) + Q_p^1(\xi_2)} \right] = J_p^I$$

and

$$\left. E_p \right|_{\mu_2 = \mu_1} = 0 \text{ from Equation (F-3)}$$

$$\text{The expression for } \left. D_p \right|_{\mu_2 = \mu_1} \text{ equals } J_p^I$$

(See Equation (42c) in the text of this report). The expression for C_p (see Equation (42a)) in the text of this report is:

$$C_p = J_p^{II} + E_p[z] \quad (F-6)$$

The expression for C_p when $\mu_2 = \mu_1$ can be expressed as:

$$\left. C_p \right|_{\mu_2 = \mu_1} = \lim_{[x] \rightarrow \infty} \left[\left. E_p \right|_{\mu_2 = \mu_1} \right] [z] + \lim_{[x] \rightarrow \infty} \left[\frac{J_{p,p}^{IP1}(\xi_2)}{[x]P_p^1(\xi_2) + Q_p^1(\xi_2)} \right]$$

$$\left. C_p \right|_{\mu_2 = \mu_1} = 0 \quad (F-7)$$

where

$$\lim_{[x] \rightarrow \infty} [z] = 0$$

and

$$\lim_{[x] \rightarrow \infty} J_p^{II} = 0$$

The expression for A_p is (see Equation (42d) in the text)

$$A_p = B_p + C_p \frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} \quad (F-8)$$

The expressions for A_p when $\mu_2 = \mu_1$ is

$$A_p \Big|_{\mu_2 = \mu_1} = B_p \Big|_{\mu_2 = \mu_1} + \left[C_p \Big|_{\mu_2 = \mu_1} \right] \frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} \quad (F-9)$$

$$A_p \Big|_{\mu_2 = \mu_1} = B_p \Big|_{\mu_2 = \mu_1} = J_p^I$$

$$\text{where } C_p \Big|_{\mu_2 = \mu_1} = 0$$

The expression for F_p (see Equation (42e) in the text) is

$$F_p = \frac{D_p P_p^1(\xi_3)}{Q_p^1(\xi_3)} + E_p \quad (F-10)$$

The expression for F_p when $\mu_2 = \mu_1$ is

$$F_p \Big|_{\mu_2 = \mu_1} = \left[D_p \Big|_{\mu_2 = \mu_1} \right] \frac{P_p^1(\xi_3)}{Q_p^1(\xi_3)} + E_p \Big|_{\mu_2 = \mu_1}$$

$$F_p \Big|_{\mu_2=\mu_1} = \left[\frac{\mu_1 J_p a (\xi_3^2 - v^2)^{1/2}}{P_p^\Delta(\xi_3) P_p^1(v)} \right] / \left[\frac{Q_p^1(\xi_3)}{P_p^1(\xi_3)} - \frac{Q_p^\Delta(\xi_3)}{P_p^\Delta(\xi_3)} \right] \quad (F-11)$$

where $E_p \Big|_{\mu_2=\mu_1} = 0$ from Equation (F-3)

This means that in the four regions, the potential used in Equation (29) of the main text of this report reduce when $\mu_2=\mu_1$ to the form

$$A_{\psi I, II, III} = \sum_{p=1}^{\infty} \left[A_p \Big|_{\mu_2=\mu_1} P_p^1(\xi) P_p^1(v) \right] \quad (F-12a)$$

$$A_{\psi IV} = \sum_{p=1}^{\infty} \left[F_p \Big|_{\mu_2=\mu_1} Q_p^1(\xi) P_p(v) \right]$$

These are the solutions for the potentials of the current band in a region of space with homogeneous permeability μ_1 (see Equations (E-1a) and (E-2a)). We now have the solution for the two potentials in regions I and II ($A_{\psi I, II, III}$, and $A_{\psi IV}$, respectively) for the simple current band problem. This indicates that the form of the coefficients A_p through F_p is correct.

The mathematical expressions for A_p and F_p which has been evaluated in the limit as $\mu_2=\mu_2$, will be compared with the coefficients A_p and F_p , respectively, for the two regions of the current band problem (see Appendix E).

A_p from Equation (F-9) is

$$A_p \Big|_{\mu_2=\mu_1} = J_p^I \quad (F-13)$$

A_p from Equation (E-6a) which we will denote here by A'_p is

$$A'_p = \left\{ \frac{\mu_{1p} J_p a (\xi_1^2 - v^2)^{1/2}}{P_p^\Delta(\xi_1) P_p^1(v)} \right\} \left[\frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} \right] \bigg/ \left[\frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} - \frac{Q_p^\Delta(\xi_1)}{P_p^\Delta(\xi_1)} \right] \quad (F-14)$$

By simple algebraic manipulation, the identity

$$\frac{1}{Q_p^\Delta(\xi_1)} \left[\frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} - \frac{Q_p^\Delta(\xi_1)}{P_p^\Delta(\xi_1)} \right] = \frac{1}{P_p^\Delta(\xi_1)} \left[\frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} \right] \left[\frac{P_p^\Delta(\xi_1)}{Q_p^\Delta(\xi_1)} - \frac{P_p^1(\xi_1)}{Q_p^1(\xi_1)} \right] \quad (F-15)$$

can be changed to read

$$\left[\frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} - \frac{Q_p^\Delta(\xi_1)}{P_p^\Delta(\xi_1)} \right] \left[\frac{P_p^1(\xi_1)}{Q_p^1(\xi_1)} \right] = - \left[\frac{Q_p^\Delta(\xi_1)}{P_p^\Delta(\xi_1)} \right] \left[\frac{P_p^1(\xi_1)}{Q_p^1(\xi_1)} - \frac{P_p^\Delta(\xi_1)}{Q_p^\Delta(\xi_1)} \right] \quad (F-16)$$

substituting Equation (F-16) into (F-14) we have

$$A'_p = \left\{ \frac{-\mu_{1p} J_p a (\xi_1^2 - v^2)^{1/2}}{Q_p^\Delta(\xi_1) P_p^1(v)} \right\} \bigg/ \left[\frac{P_p^1(\xi_1)}{Q_p^1(\xi_1)} - \frac{P_p^\Delta(\xi_1)}{Q_p^\Delta(\xi_1)} \right] \quad (F-17)$$

comparing Equation (F-17) to (41e) shows that

$$A'_p = J_p^I$$

Now F_p from Equation (E-6b) which will be denoted by F'_p is

$$F'_p = \left[\frac{\mu_1 J_p a (\xi_1^2 - v^2)^{1/2}}{P_p^\Delta(\xi_1) P_p^1(v)} \right] / \left[\frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} - \frac{Q_p^\Delta(\xi_1)}{P_p^\Delta(\xi_1)} \right] \quad (F-18)$$

which is identical to F_p as defined in Equation (F-11) so that $F_p = F'_p$

Thus, the mathematical expressions for A_p and F_p (Equations (F-13) and (F-11), respectively) for the ferromagnetic spheroidal shell surrounded by a thin current band in the limit as $\mu_2 = \mu_1$ are the same as the coefficients A_p and F_p (see Equations (E-6a) and (E-6b), respectively) for the vector potentials in the regions of the current band in free space (see Appendix E).

It is noted that when making the comparison, ξ_3 must be set equal to ξ_1 . For comparison, the coefficients for the current band problem are

$$A_p = F_p \frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} = J_p^I \quad (F-19)$$

$$F_p = \left[\frac{\mu_1 J_p a (\xi_1^2 - v^2)^{1/2}}{P_p^\Delta(\xi_1) P_p^1(v)} \right] / \left[\frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} - \frac{Q_p^\Delta(\xi_1)}{P_p^\Delta(\xi_1)} \right] \quad (F-20)$$

and the coefficients for the ferromagnetic shell problem with $\mu_2 = \mu_1$ are

$$A_p = J_p^I = F_p \frac{Q_p^1(\xi_3)}{P_p^1(\xi_3)} \quad (F-21)$$

$$F_P = \left[\frac{\mu_{1P}^J a (\xi_3^2 - v^2)^{1/2}}{P_P^\Delta(\xi_3) P_P^1(v)} \right] / \left[\frac{Q_P^1(\xi_3)}{P_P^1(\xi_3)} - \frac{Q_P^\Delta(\xi_3)}{P_P^\Delta(\xi_3)} \right] \quad (F-22)$$

APPENDIX G

REDUCTION OF THE MAGNETIC VECTOR POTENTIAL FOR A THIN COIL INTERNAL TO A FERROMAGNETIC PROLATE SPHEROIDAL SHELL TO THAT OF A THIN COIL IN FREE SPACE WHEN IN THE LIMIT μ_2 EQUALS μ_1

In this appendix, the coefficients H_p , I_p , K_p , L_p , M_p , and N_p for the potentials are evaluated for the system consisting of an infinitely thin current band surrounded by a ferromagnetic shell with permeability μ_2 in a homogeneous medium with permeability μ_1 in the limit as $\mu_2 = \mu_1$. These coefficients are utilized in Equation (47) in the section of this report entitled "Thin Coil Internal to a Ferromagnetic Prolate Spheroidal Shell." The variables are defined in Figure 3 located on page 23 of this report. When μ_2 is set equal to μ_1 the problem reduces to that of finding the potential in the two regions of a simple current band (see Figure 1-E in Appendix E), since the ferromagnetic shell will now have a permeability μ_1 equal to that of the homogeneous medium with permeability μ_1 .

In this limit the coefficients should assume the following form

$$K_p = M_p = N_p \quad (G-1a)$$

$$I_p = L_p = 0 \quad (G-1b)$$

and where H_p and N_p should reduce to the coefficients for the potentials in the two regions for the spheroidal band problem (see Appendix E). If the coefficients assume this mathematical form it will prove that the coefficients for the current band surrounded by a spheroidal shell are correct.

The mathematical solution for L_p in terms of known quantities was derived in Appendix D and was reported in the text of this report (see Equation (53)).

$$L_p = \frac{\left(\frac{1}{\mu_1}\right) \left[J_p^{II} P_p^\Delta(\xi_2) + J_p^I Q_p^\Delta(\xi_2) \right]}{\left(-\frac{1}{\mu_1}\right) [v] P_p^\Delta(\xi_2) + \left(\frac{1}{\mu_2}\right) P_p^\Delta(\xi_2) + \left(\frac{1}{\mu_2}\right) [u] Q_p^\Delta(\xi_2)} \quad (G-2a)$$

where

$$[u] = \frac{\left[\frac{P_p^1(\xi_3)}{Q_p^1(\xi_3)} - \frac{P_p^\Delta(\xi_3)}{Q_p^\Delta(\xi_3)} \left(\frac{\mu_1}{\mu_2} \right) \right]}{\left[\frac{\mu_1}{\mu_2} - 1 \right]} \quad (G-2b)$$

$$[v] = \frac{\left[P_p^1(\xi_2) + [u] Q_p^1(\xi_2) \right]}{P_p^1(\xi_2)} \quad (G-2c)$$

$$J_p = \frac{k G P_p^1(v)}{a \left(\xi_1^2 - v^2 \right)^{1/2}} \quad (G-2d)$$

$$J_p^I = \left[\frac{\mu_1 J_p a \left(\xi_1^2 - v^2 \right)^{1/2}}{P_p^\Delta(\xi_1) P_p^1(v)} \right] / \left[\frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} - \frac{Q_p^\Delta(\xi_1)}{P_p^\Delta(\xi_1)} \right] \quad (G-2e)$$

$$J_p^{II} = - J_p^I \frac{Q_p^1(\xi_2)}{P_p^1(\xi_2)} \quad (G-2f)$$

The coefficient L_p will now be evaluated when the limit is taken with $\mu_2 = \mu_1$ which causes $[u]$ and $[v]$ to approach infinity (∞).

$$L_p \Big|_{\mu_2 = \mu_1} = \lim_{\substack{[u] \rightarrow \infty \\ [v] \rightarrow \infty}} \left\{ \frac{\frac{1}{\mu_1} \left[J_p^{II} P_p^\Delta(\xi_2) + J_p^I Q_p^\Delta(\xi_2) \right]}{\left(-\frac{1}{\mu_1} \right) [v] P_p^\Delta(\xi_2) + \left(\frac{1}{\mu_1} \right) P_p^\Delta(\xi_2) + \left(\frac{1}{\mu_1} \right) [u] Q_p^\Delta(\xi_2)} \right\}$$

$$L_p \Big|_{\mu_2 = \mu_1} = 0 \quad (G-3)$$

where

$$\lim_{\mu_2 = \mu_1} [u] = \frac{\left[\frac{P_p^1(\xi_3)}{Q_p^1(\xi_3)} - \frac{P_p^\Delta(\xi_3)}{Q_p^\Delta(\xi_3)} \left(\frac{\mu_1}{\mu_1} \right) \right]}{\left[\frac{\mu_1}{\mu_1} - 1 \right]} = \infty \quad (G-4)$$

$$\lim_{\mu_2 = \mu_1} [v] = \frac{\left[P_p^1(\xi_2) + [u] Q_p^1(\xi_2) \right]}{P_p^1(\xi_2)} = \infty \quad (G-5)$$

The expression for I_p (see Equations (61) and (55))

$$I_p = J_p^{II} + L_p[v] \quad (G-6)$$

$$I_p \Big|_{\mu_2 = \mu_1} = J_p^{II} + \lim_{\mu_2 = \mu_1} [L_p][v] \quad (G-7)$$

where $\lim_{\mu_2 = \mu_1} L_p[v] =$

$$\lim_{[u] \rightarrow \infty} \frac{\left(\frac{1}{\mu_1} \right) \left[-J_p^I \frac{Q_p^1(\xi_2)}{P_p^1(\xi_2)} P_p^\Delta(\xi_2) + J_p^I Q_p^\Delta(\xi_2) \right] \left[\frac{P_p^1(\xi_2)}{P_p^1(\xi_2)} + \frac{[u] Q_p^1(\xi_2)}{P_p^1(\xi_2)} \right]}{\left(-\frac{1}{\mu_1} \right) \left[\frac{P_p^1(\xi_2)}{P_p^1(\xi_2)} + \frac{[u] Q_p^1(\xi_2)}{P_p^1(\xi_2)} \right] P_p^\Delta(\xi_2) + \frac{1}{\mu_1} P_p^\Delta(\xi_2) + \left(\frac{1}{\mu_1} \right) [u] Q_p^\Delta(\xi_2)} = -J_p^{II} \quad (G-8)$$

Thus

$$\left. I_p \right|_{\mu_2=\mu_1} = J_p^{II} - J_p^{II} \quad (G-9)$$

$$\left. I_p \right|_{\mu_2=\mu_1} = 0 \quad (G-10)$$

Now, the expression for M_p when μ_2 equals μ_1 can be written as (see Equation 60)

$$\left. M_p \right|_{\mu_2=\mu_1} = \lim_{\mu_2=\mu_1} L_p[u] \quad (G-11a)$$

where $\lim_{\mu_2=\mu_1} L_p[u] =$

$$\lim_{[u] \rightarrow \infty} \frac{\left(\frac{1}{\mu_1} \right) \left[-J_p^I \frac{Q_p^1(\xi_2)}{P_p^1(\xi_2)} P_p^\Delta(\xi_2) + J_p^I Q_p(\xi_2) \right] [u]}{\left\{ \left(-\frac{1}{\mu_1} \right) \left[\frac{P_p^1(\xi_2)}{P_p^1(\xi_2)} P_p^\Delta(\xi_2) + \frac{[u] Q_p^1(\xi_2)}{P_p^1(\xi_2)} P_p^\Delta(\xi_2) \right] + \left(\frac{1}{\mu_1} \right) P_p^\Delta(\xi_2) + \frac{1}{\mu_1} [u] Q_p^\Delta(\xi_2) \right\}} = \quad (G-11b)$$

$$= \frac{\mu_1 J_p(\theta) (\xi_1^2 - v^2)^{1/2}}{P_p^\Delta(\xi_1) P_p^1(v)} \cdot \frac{1}{\left[\frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} - \frac{Q_p^\Delta(\xi_1)}{P_p^\Delta(\xi_1)} \right]} = J_p^I$$

$$M_p \Big|_{\mu_2=\mu_1} = J_p^I \quad (G-12)$$

It is noted that this expression is identical to F_p of Equation (E-6b).

The coefficient N_p is expressed as (see Equation 62)

$$N_p = L_p \frac{P_p^1(\xi_3)}{Q_p^1(\xi_3)} + M_p \quad (G-13)$$

and since L_p was shown to be zero in Equation (G-3), N_p is equal to M_p .
The expression for K_p is defined in Equation (59) to be

$$K_p = J_p^I \quad (G-14)$$

which is equal to N_p and M_p .

The last coefficient is H_p which can be expressed as (see Equation 63)

$$H_p = I_p + K_p \frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} \quad (G-14)$$

$$H_p \Big|_{\mu_2=\mu_1} = \left[\frac{\mu_1 J_p(\theta) a(\xi_1^2 - v^2)^{1/2}}{P_p^\Delta(\xi_1) P_p^1(v)} \right] \left[\frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} \right] / \left[\frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} - \frac{Q_p^\Delta(\xi_1)}{P_p^\Delta(\xi_1)} \right] \quad (G-16)$$

This is identical with the expression for A_p in Equation (E-6a) of Appendix E.

Thus, the mathematical expressions for H_p and N_p (Equations (G-16) and (G-13), respectively) for the thin current band surrounded by a spheroidal shell in the limit as $\mu_2 = \mu_1$ are the same as the coefficients A_p and F_p (see Equations (E-6a) and (E-6b), respectively) for the vector potential in the regions of the current band in free space (see Appendix E).

For comparison, the coefficients for the ferromagnetic shell problem with $\mu_2 = \mu_1$ are

$$\left. M_p \right|_{\mu_2 = \mu_1} = J_p^I = \left[\frac{\mu_1 J_p(\theta) a (\xi_1^2 - v^2)^{1/2}}{P_p^\Delta(\xi_1) P_p(v)} \right] / \left[\frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} - \frac{Q_p^\Delta(\xi_1)}{P_p^\Delta(\xi_1)} \right] \quad (G-17)$$

$$\left. H_p \right|_{\mu_2 = \mu_1} = J_p^I \frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} = \left[\frac{\mu_1 J_p(\theta) a (\xi_1^2 - v^2)^{1/2}}{P_p^\Delta(\xi_1) P_p^1(v)} \right] \left[\frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} \right] / \left[\frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} - \frac{Q_p^\Delta(\xi_1)}{P_p^\Delta(\xi_1)} \right] \quad (G-18)$$

and the coefficients for the current band problem are

$$A_p = F_p \frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} = \left[\frac{\mu_1 J_p(\theta) a (\xi_1^2 - v^2)^{1/2}}{P_p^\Delta(\xi_1) P_p^1(v)} \right] \left[\frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} \right] / \left[\frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} - \frac{Q_p^\Delta(\xi_1)}{P_p^\Delta(\xi_1)} \right] \quad (G-19)$$

$$F_p = J_p^I = \left[\frac{\mu_1 J_p(\theta) a (\xi_1^2 - v^2)^{1/2}}{P_p^\Delta(\xi_1) P_p^1(v)} \right] / \left[\frac{Q_p^1(\xi_1)}{P_p^1(\xi_1)} - \frac{Q_p^\Delta(\xi_1)}{P_p^\Delta(\xi_1)} \right] \quad (G-20)$$

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